Implementing CreditRisk$^+$ in R Using the Fast Fourier Transform

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The CreditRisk\textsuperscript{+} Actuarial Model
Computing Loss Distribution with FFT
What Is Credit Risk?

“What Credit risk is the risk that the value of a portfolio changes due to unexpected changes in the credit quality of issuers or trading partners. This subsumes both losses due to defaults and losses due to downgradings of obligors in a rating system.”

obligor = a counterparty who has a financial obligation to us; for example, a debtor who owes us money, a bond issuer who promises interest, or a counterparty in an OTC derivatives transaction.

default = failure to fulfil that obligation, for example, failure to repay loan or pay interest/coupon on a loan/bond; generally due to lack of liquidity or insolvency; may entail bankruptcy.
Overview

1. The CreditRisk\(^+\) Actuarial Model
   - Structure of Model
   - Relation to Bernoulli Mixture Models
   - The Exposure Band Concept
   - Compound Distributions

2. Computing Loss Distribution with FFT
   - Theory
   - Practice
   - Note on Calibration
The CreditRisk+ Actuarial Model

Structure of Model

Relation to Bernoulli Mixture Models

The Exposure Band Concept

Compound Distributions
Definition of CreditRisk$^+$

- Let $\tilde{Y}_1, \ldots, \tilde{Y}_m$ denote the number of defaults for each obligor $i = 1, \ldots, m$ in a fixed time interval. (Multiple defaults are allowed but have small probability.)
- Assume that, conditional on factors $\psi = \Psi$, the variables $\tilde{Y}_1, \ldots, \tilde{Y}_m$ are independent Poisson.
- Assume that, conditional on $\Psi = \psi$, $\tilde{Y}_i \sim \text{Poi}(\lambda_i(\psi))$ where
  \[
  \lambda_i(\psi) = k_i w_i' \psi,
  \]
  for $k_i > 0$ and weight vectors $w_i = (w_{i1}, \ldots, w_{ip})'$ satisfying $\sum_{j=1}^{p} w_{ij} = 1$.
- The factors $\Psi = (\psi_1, \ldots, \psi_p)'$ are independent.
- $\psi_j \sim \text{Ga}(\sigma_j^{-2}, \sigma_j^{-2})$ for some value $\sigma_j^2$ which is the variance of $\Psi_j$. 

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Commentary on CreditRisk$^+$

- Often referred to as an actuarial model.
- Gamma mixtures of Poissons are quite widely used to model numbers of losses in non-life insurance.
- We can compute unconditional distributions relatively easily.
- Let $\tilde{M} = \sum_{i=1}^{m} \tilde{Y}_i$ denote the total number of defaults. We can show that the distribution of $\tilde{M}$ is a sum (convolution) of independent negative binomial distributions:

$$\tilde{M} \overset{d}{=} \sum_{j=1}^{p} \tilde{M}_j,$$

$$\tilde{M}_j \sim \text{NB} \left( \sigma_j^{-2}, \frac{1}{1 + \sigma_j^2 \sum_{i=1}^{m} w_{ij} k_i} \right).$$
Assume

1. $N \mid \Lambda = \lambda \sim \text{Poi}(\lambda)$
2. $\Lambda \sim \text{Ga}(\alpha, \beta)$

Then $N \sim \text{NB}(\alpha, \beta/(\beta + 1))$, a negative binomial distribution. Writing $p = \beta/(\beta + 1)$ the pmf is

$$P(N = k) = \binom{\alpha + k - 1}{k} p^\alpha (1 - p)^k, \quad \alpha > 0, 0 < p < 1,$$

where, for $x \in \mathbb{R}$ and $k \in \{0, 1, 2, \ldots\}$,

$$\binom{x}{k} = \frac{x(x - 1) \cdots (x - k + 1)}{k!}$$

is the extended binomial coefficient.
The CreditRisk$^+$ Actuarial Model

1. Structure of Model
2. Relation to Bernoulli Mixture Models
3. The Exposure Band Concept
4. Compound Distributions
General Form of Bernoulli Mixtures

- These are examples of reduced form models in which a simple statistical approach to dependent defaults is taken.
- **Definition.** Given some $p < m$ and a $p$-dimensional random vector $\psi = (\psi_1, \ldots, \psi_p)'$, the default indicator vector $Y$ follows a Bernoulli mixture model with factor vector $\psi$ if there are functions $p_i : \mathbb{R}^p \rightarrow (0, 1)$, such that conditional on $\psi$ the components of $Y$ are independent Bernoulli rvs with $P(Y_i = 1 \mid \psi = \psi) = p_i(\psi)$.
- The crucial assumption is that of conditional independence given factors, which makes these models relatively easy to analyze.
- The multivariate Merton Model, which underlies the Creditmetrics and Moody’s portfolio credit risk solutions, implies a Bernoulli mixture models for default events.
CreditRisk$^+$ as a Bernoulli Mixture Model

- Let $Y_i = I_{\{\tilde{Y}_i > 0\}}$ for $i = 1, \ldots, m$.
- The default indicators $Y_1, \ldots, Y_m$ follow a Bernoulli mixture model with factor vector $\Psi$.
- The conditional probabilities of default satisfy
  \[ p_i(\psi) = 1 - \exp(-k_i w'_i \psi). \]
- The unconditional probability of default satisfies
  \[ p_i = E(p_i(\psi)) = E(1 - \exp(-k_i w'_i \psi)) \approx k_i E(w'_i \Psi) = k_i. \]
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The Exposure Band Concept

- For $i = 1, \ldots, m$ assume that obligor losses take the form $L_i = e_i \tilde{Y}_i$. LGDs will not be considered.
- For the exposures assume that, for all $i$, we have $e_i = \ell_i \epsilon$ where $\epsilon$ is a **basic exposure size** and $\ell_i$ is a positive integer multiplier. Clearly this is an approximation in reality.
- Now define exposure bands $b = 1, \ldots, n$ corresponding to the distinct values for the multipliers $\ell^{(1)}, \ldots, \ell^{(n)}$. Let $i \in s_b$ if $\ell_i = \ell^{(b)}$. Thus $s_b$ is the set of indices for the obligors in exposure band $b$.
- Let $L^{(b)} = \sum_{i \in s_b} e_i \tilde{Y}_i = \epsilon l^{(b)} M^{(b)}$ where $M^{(b)} = \sum_{i \in s_b} \tilde{Y}_i$. (Losses in a band)
- Let $L = \sum_{b=1}^{n} L^{(b)}$ denote total loss and $M = \sum_{b=1}^{n} M^{(b)}$ denote total number of defaults as before.
The CreditRisk+ Actuarial Model

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Case of Independent Default Counts

- Suppose $\tilde{Y}_i \sim \text{Poi}(\lambda_i)$ and obligors default independently.
- In other words, we switch off the gamma-distributed factors (temporarily).
- Then $M^{(b)} \sim \text{Poi}(\lambda^{(b)})$ where $\lambda^{(b)} = \sum_{i \in s_b} \lambda_i$ is the default rate in exposure band $j$.
- Moreover $M \sim \text{Poi}(\lambda)$ where $\lambda = \sum_{b=1}^{n} \lambda^{(b)} = \sum_{i=1}^{m} \lambda_i$ is the default rate in the whole portfolio.
- What are the distributions of $L^{(b)}$ and $L$?
- Answer: compound Poisson distributions.
In general a compound Poisson random variable takes the form
\[ Z = \sum_{i=1}^{N} X_i \] where \( N \sim \text{Poi}(\mu) \) and \( X_1, X_2, \ldots \), are iid variables with distribution function \( G \).

We write \( Z \sim \text{CPoi}(\mu, G) \).

The aggregate loss in a version of CreditRisk\(^+\) with fully independent defaults has distribution

\[ L \sim \text{CPoi}(\lambda, G) \]

where \( G \) is the df of a multinomial distribution.

Under \( G \) the severity distribution of the default event is \( \epsilon \ell(b) \) with probability \( \lambda(b)/\lambda \) for \( b = 1, \ldots, n \).
In general a compound negative binomial random variable takes the form \( Z = \sum_{i=1}^{N} X_i \) where \( N \sim \text{NB}(\alpha, p) \) and \( X_1, X_2, \ldots \), are iid variables with distribution function \( G \).

We write \( Z \sim \text{CNB}(\alpha, p, G) \).

The aggregate loss in a version of CreditRisk\(^+\) with a single gamma-distributed factor is

\[
L \sim \text{CNB} \left( \sigma^{-2}, \frac{1}{1 + \sigma^2 \sum_{i=1}^{m} k_i}, G \right)
\]

where \( G \) is the df of a multinomial distribution.

Under \( G \) the severity distribution of the default event is \( \epsilon \ell(b) \) with probability \( \sum_{i \in s_b} k_i / \sum_{i=1}^{m} k_i \) for \( b = 1, \ldots, n \).

This is fairly straightforward to demonstrate using moment generating/characteristic function.
The General Case

- The loss has structure $L \overset{d}{=} \sum_{j=1}^{p} L_j$ for independent variables $L_j$ relating to each independent factor.
- The $L_j$ have compound negative binomial distributions:

  $$L_j \sim \text{CNB} \left( \sigma_j^{-2}, \frac{1}{1 + \sigma_j^2 \sum_{i=1}^{m} k_i w_{ij}}, G_j \right)$$

- $G_j$ denotes the multinomial distribution that takes the value $\epsilon \ell^{(b)}$ with probability $\sum_{i \in s_b} k_i w_{ij} / \sum_{i=1}^{m} k_i w_{ij}$ for $b = 1, \ldots, n$. 
Computing Loss Distribution with FFT

- Theory
- Practice
- Note on Calibration
Computing the Loss Distribution

- It is possible to compute the loss distribution using Fourier inversion with the fast Fourier transform (FFT).
- To use this technique we have to first compute the characteristic function of $L$.
- Recall that the characteristic function (cf) of a random variable $Z$ is given by
  $$\phi_Z(t) = E(e^{itZ})$$
- The cf of a convolution of independent variables satisfies
  $$\phi_L(t) = E(e^{itL}) = E(e^{it \sum_{j=1}^{p} L_j}) = \prod_{j=1}^{p} \phi_{L_j}(t).$$
- To compute the cf of $L_j$ we have to be able to compute the cf of a compound distribution.
Characteristic Function of Compound Distribution

- Let $Z = \sum_{k=1}^{N} X_k$ where $N$ is a frequency distribution (Poisson, negative binomial, etc.) and the $X_k$ are iid severities.

- Let $\Pi_N(s) = E(s^N) = \sum_{n=0}^{\infty} s^n P(N = n)$ denote the probability generating function (pgf) of $N$.

- We compute

$$
\phi_Z(t) = E \left( E \left( e^{it \sum_{k=1}^{N} X_k} \mid N \right) \right) = E \left( \phi_X(t)^N \right) = \Pi_N \left( \phi_X(t) \right).
$$

- The pgf of $N \sim \text{NB}(\alpha, p)$ is

$$
\Pi_N(s) = \left( \frac{1 - (1 - p)s}{p} \right)^{-\alpha}
$$
2 Computing Loss Distribution with FFT

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Calibration

- Calibration involves determination of default rates/probabilities $k_i$ for individual obligors, factor weights $w_i$ for individual obligors and factor variances $\sigma_j^2$.
- Default probabilities usually based on rating/scoring of obligors and use of historical data.
- Factors usually given a sector interpretation (financials, technology, pharma, etc.) so that weights are known.
- Factor variances determined from estimates of default correlation derived from historical data.
- Consider, for instance, the one-factor model in which the default counts $\tilde{Y}_i$ are conditionally independent Poisson variables satisfying $\tilde{Y}_i \mid \psi = \psi \sim \text{Poi}(k_i \psi)$ where $\psi \sim \text{Ga}(\sigma^{-2}, \sigma^{-2})$. 
Default Correlation in One-Factor Model

\[ \rho(Y_i, Y_j) \approx \rho(\tilde{Y}_i, \tilde{Y}_j) = \frac{E(\tilde{Y}_i \tilde{Y}_j) - k_i k_j}{\sqrt{\text{var}(\tilde{Y}_i) \text{var}(\tilde{Y}_j)}} \]

- \[ E(\tilde{Y}_i \tilde{Y}_j) = E(E(\tilde{Y}_i \tilde{Y}_j | \psi)) = k_i k_j E(\psi^2) = k_i k_j (\sigma^2 + 1) \]
- \[ E(\tilde{Y}_i^2) = E(E(\tilde{Y}_i^2 | \psi)) = E(\text{var}(\tilde{Y}_i | \psi) + E(\tilde{Y}_i | \psi)^2) = E(k_i \psi + k_i^2 \psi^2) = k_i + k_i^2 E(\psi^2) \]
- \[ \text{var}(\tilde{Y}_i) = k_i + k_i^2 \sigma^2 \]

\[ \rho(\tilde{Y}_i, \tilde{Y}_j) = \frac{k_i k_j \sigma^2}{\sqrt{(k_i + k_i^2 \sigma^2)(k_j + k_j^2 \sigma^2)}} \]