Implementing CreditRisk^{+} in R Using the Fast Fourier Transform

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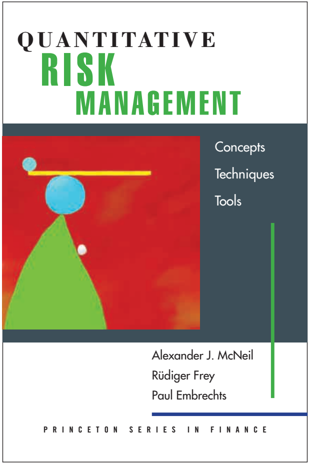
QRM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What Is Credit Risk?

"Credit risk is the risk that the value of a portfolio changes due to unexpected changes in the credit quality of issuers or trading partners. This subsumes both losses due to defaults and losses due to downgradings of obligors in a rating system."

obligor $=$ a counterparty who has a financial obligation to us; for example, a debtor who owes us money, a bond issuer who promises interest, or a counterparty in an OTC derivatives transaction.

default $=$ failure to fulfil that obligation, for example, failure to repay loan or pay interest/coupon on a loan/bond; generally due to lack of liquidity or insolvency; may entail bankruptcy.

Overview

1. The CreditRisk$^+$ Actuarial Model
   - Structure of Model
   - Relation to Bernoulli Mixture Models
   - The Exposure Band Concept
   - Compound Distributions

2. Computing Loss Distribution with FFT
   - Theory
   - Practice
   - Note on Calibration

1. The CreditRisk$^+$ Actuarial Model
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Definition of CreditRisk$^+$

- Let $\tilde{Y}_1, \ldots, \tilde{Y}_m$ denote the number of defaults for each obligor $i = 1, \ldots, m$ in a fixed time interval. (Multiple defaults are allowed but have small probability.)
- Assume that, conditional on factors $\mathbf{\Psi} = \psi$, the variables $\tilde{Y}_1, \ldots, \tilde{Y}_m$ are independent Poisson.
- Assume that, conditional on $\mathbf{\Psi} = \psi$, $\tilde{Y}_i \sim \text{Poi}(\lambda_i(\psi))$ where

$$\lambda_i(\psi) = k_i \mathbf{w}_i' \psi \ ,$$

  for $k_i > 0$ and weight vectors $\mathbf{w}_i = (w_{i1}, \ldots, w_{ip})'$ satisfying $\sum_{j=1}^p w_{ij} = 1$.
- The factors $\mathbf{\Psi} = (\Psi_1, \ldots, \Psi_p)'$ are independent.
- $\Psi_j \sim \text{Ga}(\sigma_j^{-2}, \sigma_j^{-2})$ for some value σ_j^2 which is the variance of Ψ_j.

Commentary on CreditRisk$^+$

- Often referred to an an actuarial model.
- Gamma mixtures of Poissons are quite widely used to model numbers of losses in non-life insurance.
- We can compute unconditional distributions relatively easily.
- Let $\tilde{M} = \sum_{i=1}^{m} \tilde{Y}_i$ denote the total number of defaults. We can show that the distribution of $\tilde{M}$ is a sum (convolution) of independent negative binomial distributions:

$$\tilde{M} \stackrel{d}{=} \sum_{j=1}^{p} \tilde{M}_j \ ,$$

$$\tilde{M}_j \sim \text{NB}\left(\sigma_j^{-2}, \frac{1}{1 + \sigma_j^2 \sum_{i=1}^{m} w_{ij} k_i}\right) \ .$$

Poisson-Gamma Mixtures

Assume

1. $N \mid \Lambda = \lambda \sim \text{Poi}(\lambda)$
2. $\Lambda \sim \text{Ga}(\alpha, \beta)$

Then $N \sim \text{NB}(\alpha, \beta/(\beta+1))$, a negative binomial distribution.
Writing $p = \beta/(\beta+1)$ the pmf is

$$P(N = k) = \binom{\alpha + k - 1}{k} p^{\alpha}(1-p)^k \,, \quad \alpha > 0, 0 < p < 1,$$

where, for $x \in \mathbb{R}$ and $k \in \{0, 1, 2, \ldots\}$,

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$$

is the extended binomial coefficient.

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General Form of Bernoulli Mixtures

- These are examples of reduced form models in which a simple statistical approach to dependent defaults is taken.
- Definition. Given some $p < m$ and a p-dimensional random vector $\mathbf{\Psi} = (\Psi_1, \ldots, \Psi_p)'$, the default indicator vector $\mathbf{Y}$ follows a Bernoulli mixture model with factor vector $\mathbf{\Psi}$ if there are functions $p_i : \mathbb{R}^p \to (0, 1)$, such that conditional on $\mathbf{\Psi}$ the components of $\mathbf{Y}$ are independent Bernoulli rvs with $P(Y_i = 1 \mid \mathbf{\Psi} = \psi) = p_i(\psi)$.
- The crucial assumption is that of conditional independence given factors, which makes these models relatively easy to analyze.
- The multivariate Merton Model, which underlies the Creditmetrics and Moody's portfolio credit risk solutions, implies a Bernoulli mixture models for default events.

CreditRisk$^+$ as a Bernoulli Mixture Model

- Let $Y_i = I_{\{\tilde{Y}_i > 0\}}$ for $i = 1, \ldots, m$.
- The default indicators $Y_1, \ldots, Y_m$ follow a Bernoulli mixture model with factor vector $\boldsymbol{\Psi}$.
- The conditional probabilities of default satisfy

$$p_i(\psi) = 1 - \exp(-k_i \mathbf{w}_i' \psi).$$

- The unconditional probability of default satisfies

$$\begin{aligned} p_i = E\left(p_i(\boldsymbol{\Psi})\right) &= E\left(1 - \exp(-k_i \mathbf{w}_i' \boldsymbol{\Psi})\right) \\ &\approx k_i E\left(\mathbf{w}_i' \boldsymbol{\Psi}\right) = k_i \end{aligned}$$

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The Exposure Band Concept

- For $i = 1, \dots, m$ assume that obligor losses take the form $L_i = e_i \tilde{Y}_i$. LGDs will not be considered.
- For the exposures assume that, for all i, we have $e_i = \ell_i \epsilon$ where ϵ is a basic exposure size and ℓ_i is a positive integer multiplier. Clearly this is an approximation in reality.
- Now define exposure bands $b = 1, \dots, n$ corresponding to the distinct values for the multipliers $\ell^{(1)}, \dots, \ell^{(n)}$. Let $i \in s_b$ if $\ell_i = \ell^{(b)}$. Thus s_b is the set of indices for the obligors in exposure band b.
- Let $L^{(b)} = \sum_{i \in s_b} e_i \tilde{Y}_i = \epsilon l^{(b)} M^{(b)}$ where $M^{(b)} = \sum_{i \in s_b} \tilde{Y}_i$. (Losses in a band)
- Let $L = \sum_{b=1}^{n} L^{(b)}$ denote total loss and $M = \sum_{b=1}^{n} M^{(b)}$ denote total number of defaults as before.

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Case of Independent Default Counts

- Suppose $\tilde{Y}_i \sim \text{Poi}(\lambda_i)$ and obligors default independently.
- In other words, we switch off the gamma-distributed factors (temporarily).
- Then $M^{(b)} \sim \text{Poi}(\lambda^{(b)})$ where $\lambda^{(b)} = \sum_{i \in s_b} \lambda_i$ is the default rate in exposure band j.
- Moreover $M \sim \text{Poi}(\lambda)$ where $\lambda = \sum_{b=1}^{n} \lambda^{(b)} = \sum_{i=1}^{m} \lambda_i$ is the default rate in the whole portfolio.
- What are the distributions of $L^{(b)}$ and L?
- Answer: compound Poisson distributions.

Compound Poisson

- In general a compound Poisson random variable takes the form $Z = \sum_{i=1}^{N} X_i$ where $N \sim \text{Poi}(\mu)$ and $X_1, X_2, \ldots,$ are iid variables with distribution function G.
- We write $Z \sim \text{CPoi}(\mu, G)$.
- The aggregate loss in a version of CreditRisk$^+$ with fully independent defaults has distribution

$$L \sim \text{CPoi}\,(\lambda, G)$$

  where G is the df of a multinomial distribution.
- Under G the severity distribtion of the default event is $\epsilon\ell^{(b)}$ with probability $\lambda^{(b)}/\lambda$ for $b = 1, \ldots, n$.

Compound Negative Binomial

- In general a compound negative binomial random variable takes the form $Z = \sum_{i=1}^{N} X_i$ where $N \sim \text{NB}(\alpha, p)$ and $X_1, X_2, \ldots,$ are iid variables with distribution function G.
- We write $Z \sim \text{CNB}(\alpha, p, G)$.
- The aggregate loss in a version of CreditRisk$^+$ with a single gamma-distributed factor is

$$L \sim \text{CNB}\left(\sigma^{-2}, \frac{1}{1 + \sigma^2 \sum_{i=1}^{m} k_i}, G\right)$$

  where G is the df of a multinomial distribution.
- Under G the severity distribution of the default event is $\epsilon \ell^{(b)}$ with probability $\sum_{i \in s_b} k_i / \sum_{i=1}^{m} k_i$ for $b = 1, \ldots, n$.
- This is fairly straightforward to demonstrate using moment generating/characteristic function.

The General Case

- The loss has structure $L \stackrel{d}{=} \sum_{j=1}^{p} L_j$ for independent variables L_j relating to each independent factor.
- The L_j have compound negative binomial distributions:

$$L_j \sim \text{CNB}\left(\sigma_j^{-2}, \frac{1}{1+\sigma_j^2 \sum_{i=1}^{m} k_i w_{ij}}, G_j\right)$$

- G_j denotes the multinomial distribution that takes the value $\epsilon\ell^{(b)}$ with probability $\sum_{i \in s_b} k_i w_{ij} / \sum_{i=1}^{m} k_i w_{ij}$ for $b = 1, \ldots, n$.

2 Computing Loss Distribution with FFT

- Theory
- Practice
- Note on Calibration

Computing the Loss Distribution

- It is possible to compute the loss distribution using Fourier inversion with the fast Fourier transform (FFT).
- To use this technique we have to first compute the characteristic function of L.
- Recall that the characteristic function (cf) of a random variable Z is given by

$$\phi_Z(t) = E\left(e^{itZ}\right)$$

- The cf of a convolution of independent variables satisfies

$$\phi_L(t) = E\left(e^{itL}\right) = E\left(e^{it\sum_{j=1}^{p} L_j}\right) = \prod_{j=1}^{p} \phi_{L_j}(t).$$

- To compute the cf of L_j we have to be able to compute the cf of a compound distribution.

Characteristic Function of Compound Distribution

- Let $Z = \sum_{k=1}^{N} X_k$ where N is a frequency distribution (Poisson, negative binomial, etc.) and the X_k are iid severities.
- Let $\Pi_N(s) = E(s^N) = \sum_{n=0}^{\infty} s^n P(N = n)$ denote the probability generating function (pgf) of N.
- We compute

$$\begin{aligned} \phi_Z(t) &= E\left(E\left(e^{it\sum_{k=1}^{N} X_k} \mid N\right)\right) \\ &= E\left(\phi_X(t)^N\right) = \Pi_N\left(\phi_X(t)\right). \end{aligned}$$

- The pgf of $N \sim \text{NB}(\alpha, p)$ is

$$\Pi_N(s) = \left(\frac{1-(1-p)s}{p}\right)^{-\alpha}$$

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Calibration

- Calibration involves determination of default rates/probabilities k_i for individual obligors, factor weights $\boldsymbol{w}_i$ for individual obligors and factor variances σ_j^2.
- Default probabilities usually based on rating/scoring of obligors and use of historical data.
- Factors usually given a sector interpretation (financials, technology, pharma, etc.) so that weights are known.
- Factor variances determined from estimates of default correlation derived from historical data.
- Consider, for instance, the one-factor model in which the default counts $\tilde{Y}_i$ are conditionally independent Poisson variables satisfying $\tilde{Y}_i \mid \Psi = \psi \sim \text{Poi}(k_i\psi)$ where $\Psi \sim \text{Ga}(\sigma^{-2}, \sigma^{-2})$.

Default Correlation in One-Factor Model

$$\rho(Y_i, Y_j) \approx \rho(\tilde{Y}_i, \tilde{Y}_j) = \frac{E(\tilde{Y}_i \tilde{Y}_j) - k_i k_j}{\sqrt{\operatorname{var}(\tilde{Y}_i) \operatorname{var}(\tilde{Y}_j)}}$$

- $E(\tilde{Y}_i \tilde{Y}_j) = E(E(\tilde{Y}_i \tilde{Y}_j \mid \Psi)) = k_i k_j E(\Psi^2) = k_i k_j(\sigma^2 + 1)$
- $E(\tilde{Y}_i^2) = E(E(\tilde{Y}_i^2 \mid \Psi)) = E(\operatorname{var}(\tilde{Y}_i \mid \Psi) + E(\tilde{Y}_i \mid \Psi)^2) = E(k_i \Psi + k_i^2 \Psi^2) = k_i + k_i^2 E(\Psi^2)$
- $\operatorname{var}(\tilde{Y}_i) = k_i + k_i^2 \sigma^2$

$$\rho(\tilde{Y}_i, \tilde{Y}_j) = \frac{k_i k_j \sigma^2}{\sqrt{(k_i + k_i^2 \sigma^2)(k_j + k_j^2 \sigma^2)}}$$