An R package of a partial internal model for life insurance

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Introduction

- Under Solvency II framework, in order to protect the benefit of shareholder and policyholder, the insurance company should be adequately capitalized to fulfill the capital requirement for solvency.
- Two main components should be taken into account:
  - Available capital, which refers to shareholders net asset value and is defined as the difference between the market value of assets and liabilities.
    - A stochastic cash flow projection model is used to capture the evolution of cash flows of assets and liabilities.
    - An Economic Scenario Generator (ESG) is used to generate economic scenarios including the financial market risk factors through Monte Carlo simulation.
  - Solvency Capital Requirement (SCR), which refers to the 99.5% VaR of the available capital over one year horizon.
    - The nested simulation and replicating portfolio is used to calculate the SCR.

Overview

1. Market consistent valuation of available capital
   - Stochastic cash flow projection model
   - Economic Scenario Generator

2. Risk modeling for SCR calculation
   - Solvency capital requirement
   - Replicating portfolio

3. Empirical application

Simplified balance sheet

- Simplified balance sheet at time t:

Assets	**Liabilities**
${}^{MV}A_t$	L_t
	R_t

  - ${}^{MV}A_t$ is the market value of asset portfolio ${}^{MV}A_t$.
  - L_t is the book value of policyholder's account value
  - R_t is the reserve account, which is a hybrid determined as the difference between a market value and book value, i.e. $R_t = {}^{MV}A_t - L_t$.
  - The difference between the book value of assets and liabilities is the shareholder's equity, i.e. $E_t = {}^{BV}A_t - L_t$.
- Asset model
- Liability model

Asset model

- The asset portfolio consists of coupon bonds and stocks with constant strategic asset allocation. The proportion of market value of stocks in the asset portfolio is p^{SAA}. The asset portfolio is rebalanced at the beginning of each year.
- A ratio p^{UGL} of the Unrealized Gain and Loss (UGL), i.e. the difference of market and book value, of stocks is realized when UGL is positive. If UGL of stock is negative, then 100% are realized.
- The earnings and return on book value is

$$^{BV}I_t = UGL_t^S \cdot p^{UGL} + CF_t \text{ and } {}^{BV}r_t = \frac{^{BV}I_t}{^{BV}A_{t-1}^{+} + CF_{t-1}^{P}}.$$

where UGL_t^S is the UGL of stock at t and CF_t is the realized cash flow of the asset portfolio at end of year t. $^{BV}A_{t-1}^{+}$ be the book value of asset portfolio after the in/out cash flows payments to shareholders at beginning of year t.

Liability model

- The liability portfolio consists of German traditional participating life insurance contracts (endowment assurance) with different ages and durations:
  - A minimum interest rate g is guaranteed on the actuarial reserves.
  - A minimum participation rate δ of the earnings on book values is credited to the policyholder's account.
- Assumptions:
  - The charges such as initial acquisition charge and administration charge etc are not considered.
  - The mortality rates for the premium calculation are based on DAV 2008 T (German standard mortality table).
- The policyholder's account is the sum of actuarial and bonus reserve:
  - Actuarial reserve is calculated recursively by actuarial principle of equivalence to meet the future payment of guaranteed benefit.
  - Bonus reserve consists of part of surplus to policyholder and guaranteed interest rates on bonus reserve.
- In the event of a claim, the benefit consists of bonus reserve and the guaranteed benefit is paid out to the policyholder.

Surplus distribution

- Consider only the investment surplus $Sp_t = {}^{BV}I_t^{AbL} - I_t^g$,
  - ${}^{BV}I_t^{AbL}$ is actual investment earnings on book value of assets backing liabilities
  - I_t^g is the amount credited to the policyholder account due to profit sharing and guaranteed interest rate.
- According to the German regulatory, a minimum surplus participation rate δ (based on MindZV) of the earnings on book values should be credited the policyholders' account, i.e.

$$PS_t = \max(\delta \cdot {}^{BV}I_t^{AbL} - I_t^g, 0).$$

- The surplus are assumed to be distributed such that all policyholders receive the same total yield on their account.
- If capital contribution is required then it is $c_t = \max\{L_t - {}^{MV}A_t^-, 0\}$.
- The remaining part of surplus ${}_{sh}X_t = {}^{L}X_t + {}^{RC}X_t - c_t$ goes to shareholders, which represents the in/out cash flow payment to shareholders.

Available capital at $t = 0$

- The available capital at time $t = 0$ is assumed to be calculated as [1]

$$AC_0 = \mathbb{E}^{\mathbb{Q}}\left(\sum_{t=1}^{T} \frac{X_t}{B_t}\right) \tag{1}$$

where $B_t = \exp\left(\int_0^t r_u \, du\right)$ and

$$X_t = \begin{cases} {}_{sh}X_t & \text{if } t \in 1, \ldots, T-1 \\ {}_{sh}X_t + RC_T & \text{if } t = T \end{cases} .$$

- The company's future obligations in the policyholder's account are given by:

$$^{MV}L_0 = \mathbb{E}^{\mathbb{Q}}\left(\sum_{t=1}^{T} -\frac{CF_{t-1}^{P}}{B_{t-1}} + \frac{{}_{ph}X_t}{B_t}\right) . \tag{2}$$

- If all cash fows are properly captured by the cash flow projection model, the relationship $AC_0 + {}^{MV}L_0 = {}^{MV}A_0$ holds for leakage test.

[1] Here we simply assume that the available capital, MCEV and own funds are the same.

Economic Scenario Generator

- Models in Economic Scenario Generator
  - Interest rate: Extended multi factor Cox-Ingersoll-Ross model
  - Equity: Heston model
- Monte Carlo simulation for generating economic scenarios for interest rate and equity risk factors.

Solvency Capital Requirement (SCR)

- Let AC_1 be the value of available capital at $t = 1$.
- Let L be the one-year loss function at $t = 1$

$$L := AC_0 - \frac{1}{1 + rr(0,1)} AC_1, \tag{3}$$

where $rr(0,1)$ one year risk free spot rate.

- The Solvency Capital Requirement (SCR) is defined as:

$$SCR := \text{VaR}_\alpha(L) = \inf\{x \in \mathbb{R} : \mathbb{P}[L > x] \geq 1 - \alpha\}, \tag{4}$$

where α represents the confidence level and is set to be equal to 99.5%.

Available capital at $t = 1$

- The available capital at $t = 1$ conditional on one year evolution of the financial market under real world measure, i.e.

$$
\begin{aligned}
AC_1 = V_1 &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=2}^{T} \exp\left(-\int_1^t r_u\,du\right) X_t \,\middle|\, Y_s,\ 0 \le s \le 1\right] + X_1 \\
&= \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} \exp\left(-\int_1^t r_u\,du\right) X_t \,\middle|\, Y_s,\ 0 \le s \le 1\right], \qquad (5)
\end{aligned}
$$

  and X_1 is the profit assumed to be not paid to shareholders yet.
- Assume that all necessary information for the projection of the cash flows is contained in a finite collection of Markov State Variables (Y_1, D_1) (see Bauer et al. (2009) [2]). Then

$$
V_1 = \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} \exp\left(-\int_1^t r_u\,du\right) X_t \,\middle|\, (Y_1, D_1)\right]. \qquad (6)
$$

[2] D. Bauer, D. Bergmann, and A. Reuss. Solvency II and Nested Simulations - a Least-Squares Monte Carlo Approach. Preprint Series, Ulm University. Also Availabe at http://numerik.uni-ulm.de/preprints/2009/200905_Solvency_Preprint-Server.pdf, 2009

Nested simulation and proxy approaches

- Nested simulation approach with sufficient large outer (real world) and inner (risk neutral) paths requires high computational time.
- Proxy approaches are based on finding a linear combination of basis functions to approximate the V_1.
  - Let $B_k(Y_1, D_1)$ be the k-th basis function, then the finite linear combination of basis functions VA_1 is used to approximate value of V_1, i.e.

$$V_1 \approx VA_1 = \sum_{k=1}^{M} \beta_k B_k(Y_1, D_1) \tag{7}$$

    where M is the number of basis functions.
  - Least square Monte-Carlo (LSMC): basis functions are risk factors at $t = 1$
  - Curve fitting (e.g. Delta-Gamma): basis functions are risk factors
  - Replicating portfolio: basis functions are assets

Replicating portfolio

- Given an asset pool of financial assets, find out an optimal replicating portfolio G^* and optimal weights such that the cash flow of replicating portfolio Z^G could match the cash flow X of shareholder's future profits as well as possible, i.e.

$$\min_{w^G,\ G} d(X, Z^G) \tag{8}$$

  where d is the L2-norm that measures the distance between X and Z^G.
- Criterion for judgment of the quality of replicating portfolio
  - Quality of cash flow matching (or present value matching)
  - Calibration error
  - Estimation error of SCR
  - Robustness and over fitting
  - Long short positions and offsetting effects

Empirical application

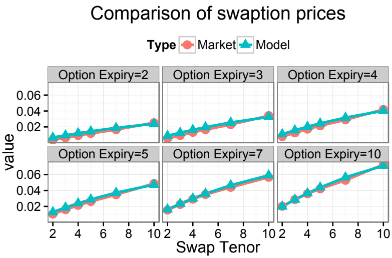
1. Market data
2. Model calibration
   - Calibration of the extended multi-factor CIR model
   - Calibration of Heston model
3. Scenario generation and validation
   - Risk neutral scenario generation and validation
   - Real world scenario generation
4. Market consistent valuation
5. SCR calculation by nested simulation and replicating portfolio

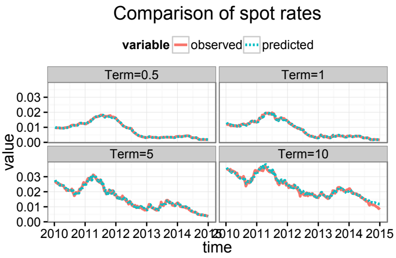
Market data from Bloomberg

- Market data at cutoff-date 31.12.2014:
  - Swap rate with different maturities.
  - ATM swaption volatilities with different option expiries and swap tenors.
  - Market prices of European options for EuroStoxx.
- Historical data from 30.06.1999 to 31.12.2014:
  - Historical swap rate with different maturities.
  - The historical data of EuroStoxx (sx5e) and the corresponding volatility index (v2x).

Calibration of interest rate model

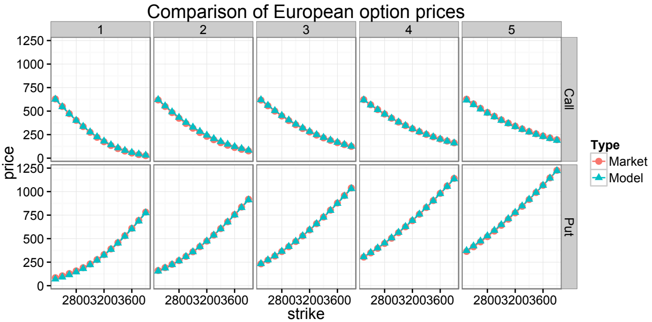
- Choose $N = 3$, i.e. the extended three factor CIR model, which is calibrated to the historical data of continuous compounded spot rates as well as the swaption prices with different option expiries and swap tenor at cutoff date.
- The target is to minimize the mean squared errors between the market swaption prices and the model swaption prices and maximize the log-likelihood of quasi maximum likelihood through Kalman filter.





Calibration of equity model

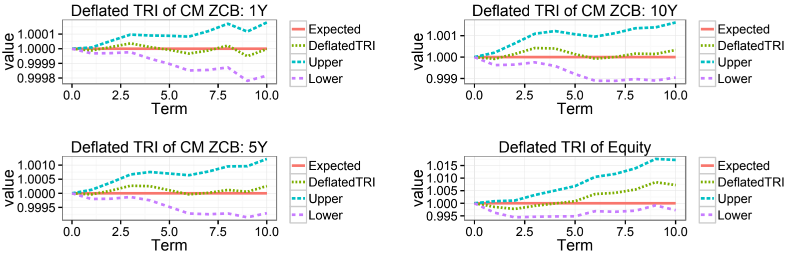
- For the real world calibration, the equity model is calibrated to the historical data of equity prices by the maximum likelihood estimation in closed-form. [3]
- For the risk neutral calibration, the equity model is calibrated to the European option prices by minimizing mean squared errors between the market and model option prices.



[3] Y. Aït-Sahalia and R. Kimmel. Maximum likelihood estimation of stochastic volatility models. *Journal of Econometrics*, 83:413–452, 2007

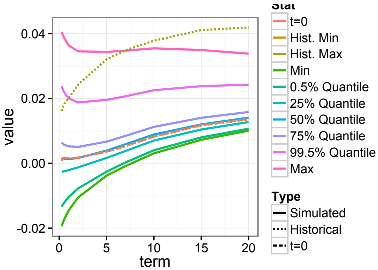
Risk neutral scenario generation and validation

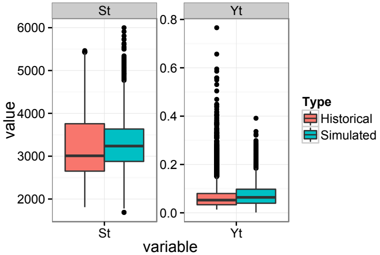
- Risk neutral scenario generation for market consistent valuation
  - Use standard Euler scheme with 250 time steps per year with simulation horizon of 10 years.
  - Antithetic Variates is used for the variance reduction.
  - Number of risk neutral simulation is 10000, i.e. 5000 pairs of antithetic scenarios.
- Validation
  - Root Mean Squared Relative Error between the model and MC based prices are 0.39% and 0.59% for swaption and equity option.
  - Martingale test of total return indices (TRI).



Real world scenario generation and validation

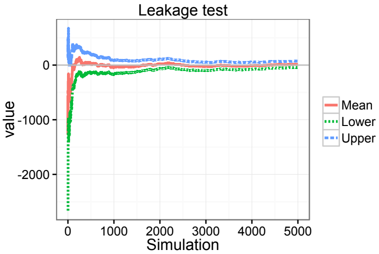
- Real world scenario generation for SCR calculation
  - Use the exact simulation of CIR process to avoid generating negative value of state variables.
  - Number of real world simulation is 10000.
- Validation
  - The proportions of variance for first three components by PCA analysis of simulated data are 92.57%, 7.20%, 0.23% (Historical: 92.74%, 6.62%, 0.47%).
  - Coverage of historical data

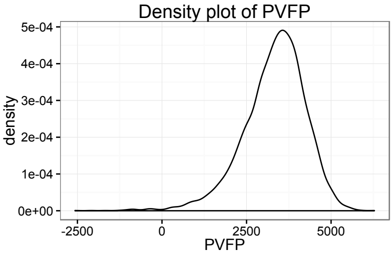




Market consistent valuation

- The parameters for asset-liability-model
  - The liability portfolio is built up at cutoff date by entering 10,000 new policyholders aged at 50 with life insurance contracts expired in 1 to 10 years proportionally. The guaranteed benefit is 20 TEUR and guaranteed rate is 0.75% for each contract.
  - For asset portfolio, $p^{SAA} = 5\%$ and $p^{UGL} = 20\%$.
  - Initial reserve is 5,000 TEUR.
- The distribution of present values of shareholder's future profits (PVFP) is left-heavy tailed. The MCEV or AC_0 by averaging these values is 3319.82 with standard error of 13.08.



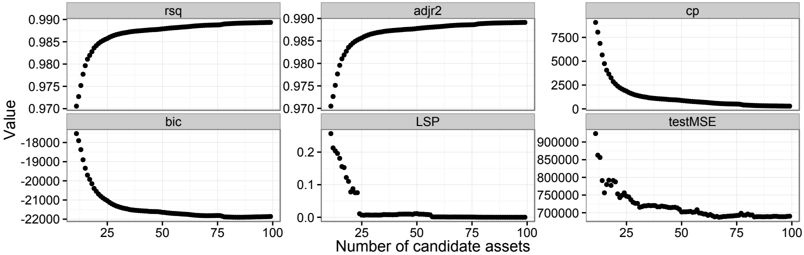


Nested simulation

- Generate $M = 10000$ outer scenarios under real world measure up to time $t = 1$.
- For each outer real world scenario i:
  - Generate $K = 1000$ inner scenarios under risk neutral measure.
  - For each inner risk neutral scenario k, compute the sum of discounted future profits $PV_1^{(i,k)}$.
  - Evaluate the PVFP at $t = 1$ conditional on outer scenario i, $\widehat{V}_1^{(i)}(K)$, by taking the average of $PV_1^{(i,k)}$ for $k = 1, \ldots, K$ over all inner risk neutral scenarios.

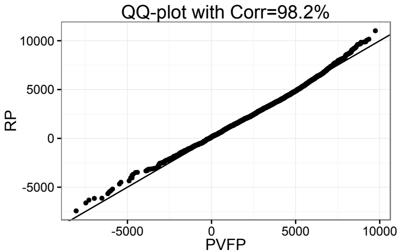
Replicating portfolio

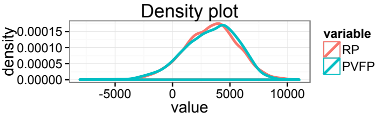
- Calibration set (in sample): 5000 risk neutral scenarios with independent random variables at cutoff date $t = 0$.
- Test set (out of sample): 100×50 risk neutral scenarios at $t = 1$, i.e. 50 inner risk neutral scenarios conditional on each of 100 outer scenarios with smallest short rates.
- Selection criterion: i.e. Multiple R-squared, adjusted R-squared, Mallows'C_p, BIC, Long-Short-Position ($LSP = \frac{|\sum_i w_i \sigma_i|}{\sum_i |w_i \sigma_i|}$) and mean squared error of test set.
- Method: forward stepwise subset selection.

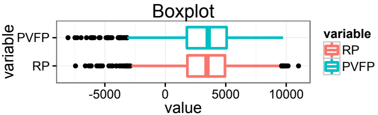


Replicating portfolio (cont.)

- Choose replicating portfolio with 62 candidate assets by the combined criteria: $\max_j \left\{ LSP(j) \,\middle|\, \left|\frac{testMSE(j)}{testMSE_{min}} - 1\right| < 0.01 \right\}$.
- Diagnostic of linear regression among PVFP and candidate assets
  - Most of coefficients for candidate assets are significant according to the t-statistics.
  - The multiple R^2 and adjusted R^2 are 98.84% and 98.83%.
  - The p-value of F-statistic is smaller than 2.2e-16.

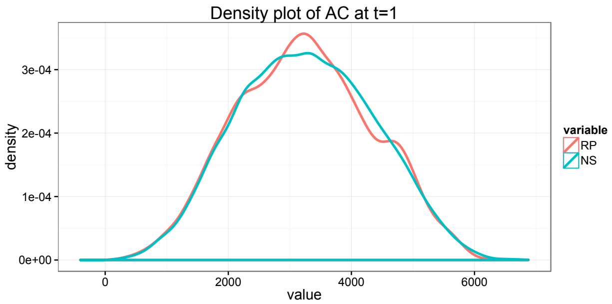






The SCR calculation

- Given the AC_0 3319.8 and one year interest rate 0.1615%, the SCR calculated by nested simulation and replicating portfolio are:
  - Nested simulation
    - The 0.5% quantile of AC_1 is 778.86.
    - the SCR is 2542.2 with solvency ratio (AC_0/SCR) 130.6%.
  - Replicating portfolio
    - The 0.5% quantile of AC_1 is 756.37.
    - the SCR is 2564.7 with solvency ratio (AC_0/SCR) 129.4%.
- Comparison of the distribution of AC_1.



Implementation in an R package

- The implementation of partial internal model is coded into an R package called rIMLife:
  - The Kalman filter algorithm, Asset-liability-model and Monte-Carlo simulation and fast computation required functions are implemented in C++ through Rcpp and RcppArmadillo packages.
  - The selection of replicating portfolio uses the method of subset selection in the leaps package.
  - Local and global optimization algorithms (e.g. differential evolution in DEoptim package) are used for model calibration.
- Data processing and visualization: plyr, ggplot2, reshape2
- Documentation: knitr
- Unit test and code optimization: testthat, profvis

Conclusion and remarks

- A simple partial internal model is constructed to illustrate the calculation of available capital (MCEV) and SCR for a life insurance company by given market data.
- The method of replicating portfolio is a good approximation instead of nested simulation.
- Based on this partial internal model, one could e.g.
  - develop new insurance products under low interest rate environment by considering the resulting MCEV and SCR.
  - construct a benchmark portfolio for validation purpose comparing to the real portfolio.
- Further extensions: credit model, life annuity and unit-link products, curve fitting and LSMC, etc.

Backup

Extended Multi-factor Cox-Ingersoll-Ross model for interest rates

- The short rate dynamics under real world measure are given by

$$r(t) = \delta(t) + \sum_{i=1}^{N} X_i(t), \tag{9}$$

$$dX_i(t) = \kappa_i'(\theta_i' - X_i(t))dt + \sigma_i \sqrt{X_i(t)} dW_i^{\mathbb{P}}(t) \tag{10}$$

  where $W_i^P(t)$ for $i = 1, \ldots, N$ are independent Wiener processes under real world measure.

- The short rate dynamics under risk neutral measure

$$dX_i(t) = \kappa_i(\theta_i - X_i(t))dt + \sigma_i \sqrt{X_i(t)} dW_i^{\mathbb{Q}}(t), \tag{11}$$

  by changing of measure with $dW_i^{\mathbb{Q}}(t) = dW_i^{\mathbb{P}}(t) + \frac{\lambda_i^0 + \lambda_i^1 X_i(t)}{\sigma_i \sqrt{X_i(t)}} dt$, where λ_i^0, λ_i^1 are the parameters for market price of risk and

$$\kappa_i' = \kappa_i - \lambda_i^1, \quad \theta_i' = \frac{\kappa_i \theta_i + \lambda_i^0}{\kappa_i - \lambda_i^1} \quad \text{for } i = 1, \ldots, N.$$

Pricing bonds and options

- Pricing zero coupon bond with

$$P(t, T) = e^{C(t,T) - \sum_{i=1}^{N} B^{X_i}(t,T) X_i(t)}.$$

- Pricing European zero coupon bond option by using Fourier Transform, when the characteristic function of $B^{X_i}(t, T)X_i(t)$ (a linear combination of non-central χ^2 random variables) is given. [4] [5]
- Pricing swaption by stochastic duration approximation, i.e. the swaption could be approximated by a multiple of the price of European zero coupon bond option. [6]

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[4] R.-R. Chen and L. Scott. Interest rate options in multifactor cox-ingersoll-ross models of the term structure. *The Journal of Derivatives*, pages 53–72, 1995

[5] P. Carr and D. B. Madan. Option valuation using the fast fourier transform. *Journal of Computational Finance*, 3:463–520, 1999

[6] C. Munk. Stochastic duration and fast coupon bond option pricing in multi-factor models. *Review of Derivatives Research*, 3:157–181, 1999

Heston model for equity

- The dynamics of $s(t) = \ln S(t)$ under real world measure:

$$ds(t) = (r(t) - q + a + bv(t))dt + \sqrt{v(t)}(\rho dW_v^{\mathbb{P}}(t) + \sqrt{1-\rho^2} dZ_s^{\mathbb{P}}(t)),$$
$$dv(t) = \kappa_v'(\theta_v' - v(t))dt + \sigma_v \sqrt{v(t)} dW_v^{\mathbb{P}}(t),$$

with $dW_v^{\mathbb{P}}(t) dZ_s^{\mathbb{P}}(t) = 0$

- The dynamics of $s(t) = \ln S(t)$ under risk neutral measure:

$$ds(t) = (r(t) - q - \frac{1}{2}v(t))dt + \sqrt{v(t)} \left(\rho dW_v^{\mathbb{Q}}(t) + \sqrt{1-\rho^2} dZ_s^{\mathbb{Q}}(t)\right),$$
$$dv(t) = \kappa_v(\theta_v - v(t))dt + \sigma_v \sqrt{v(t)} dW_v^{\mathbb{Q}}(t)$$

by changing the measure $dZ_s^{\mathbb{Q}}(t) = dZ_s^{\mathbb{P}}(t) + \frac{\lambda_s^0 + \lambda_s^1 v(t)}{\sqrt{1-\rho^2}\sqrt{v(t)}} dt$ and $dW_v^{\mathbb{Q}}(t) = dW_v^{\mathbb{P}}(t) + \frac{\lambda_v^0 + \lambda_v^1 v(t)}{\sigma_v \sqrt{v(t)}} dt$, where $a = \frac{\rho}{\sigma_v}\lambda_v^0 + \lambda_s^0$, $b = \frac{\rho}{\sigma_v}\lambda_v^1 + \lambda_s^1 - \frac{1}{2}$, $\kappa_v' = \kappa_v - \lambda_v^1$, $\theta_v' = \frac{\kappa_v \theta_v + \lambda_v^0}{\kappa_v - \lambda_v^1}$.

Pricing European option on equity

- Let $C(t; T, K) = \mathbb{E}^{\mathbb{Q}}[\exp(-\int_t^T r(t)dt)(S_T - K)^+]$ be the price of European call option with maturity T and strike K at time t.
- The price of European call option could be calculated by Fourier transformation. [7]

$$C(t; K, T) = \frac{\exp\{-a \ln K\}}{\pi} \int_0^\infty Re[\exp\{-iu \ln K\}\zeta_c(u; t, T)]du,$$

where $\zeta_c(u; t, T) = \frac{\phi(u-(a+1)i, \boldsymbol{X}(t), T-t)}{a^2+a-u^2+i(2a+1)u}$ and $\phi(u; \boldsymbol{X}(t), t, T)$ is the discounted characteristic function of $s(t)$ for state vector $\boldsymbol{X}(t) = [S(t), v(t), X_1(t), \ldots, X_N(t)]^T$. It has closed form by solving Riccati ODEs. [8]

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[7] P. Carr and D. B. Madan. Option valuation using the fast fourier transform. *Journal of Computational Finance*, 3:463–520, 1999

[8] L. A. Grzelak and C. W. Oosterlee. On the hestion model with stochastic interest rate. *SIAM J. Financial Math.*, 2:255–286, 2011

Pool of financial assets

- The pool of financial assets include different types of assets (traded and synthetic assets) that could replicate the cash flows of future profits.
- Interest rate related assets
  - Risk free zero coupon bonds with different maturities
  - Total return indices of zero coupon bond
  - Total return indices of constant maturity zero coupon bond
  - Interest rate swaps (Receiver) with different strikes
  - Receiver swaptions with different option expiries and swap tenors
- Equity related assets
  - Total return indices of equity at different years
  - European put options with different option maturities and strikes