Structural sensitivity effects

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Motivation

- Several importance measures have been defined to understand the structure of complex models, for instance:
  - Moment-independent measures (Borgonovo, 2007);
  - High-dimensional model representation (Li et al., 2010);
  - Shapley values (Owen, 2014);
  - Stress-based indices (Pesenti, Millossovich and Tsanakas, 2019)
- However, most indices include the impact of both model structure and input dependence.
  - Consider the model

    $$f(X_1, X_2) = X_1 \cdot X_2$$

    with $corr(X_2, X_3) \neq 0$.
  - $Imp(X_3) \neq 0$ but f does not contain X_3.
- We want to consider indices for the structural contribution of inputs.

Structural and correlative sensitivity indices (Li et al., 2010)

- Consider the finite hierarchical expansion of $Y = f(X)$ as

$$f(\mathrm{X}) = \sum_{u \subseteq N} f_u(X_u) = \sum_{i=1}^{d} f_i(X_i) + \sum_{i<j} f_{i,j}(X_{i,j}) + ... + f_N(X_N),$$

  where X_u are the components of X indexed by $u \subseteq N = \{1, 2, ..., d\}$.

- The covariance decomposition of the variance of the output is

$$\begin{aligned}\mathbb{V}[Y] &= \sum\nolimits_{\emptyset \neq u \subseteq N} \left[\mathbb{V}[f_u(X_u)] + \mathrm{Cov}\left(f_u(X_u), \sum\nolimits_{\emptyset \neq v \subseteq N, v \neq u} f_v(X_v)\right)\right] \\ &= \sum\nolimits_{\neq u \subset N} [V_u + V_u^c]\end{aligned}$$

  where V_u is the structural and V_u^c the correlative contribution.

Example: Linear Gaussian model

Consider the linear Gaussian model

$$Y = 2X_1 + 3X_2$$

where the inputs have a standard multivariate normal distribution with correlation coefficient ρ.

ρ	$\mathbb{V}[Y]$	V_1	V_1^c	V_2	V_2^c	$V_{1,2}$	$V_{1,2}^c$
0.7	21.4	4	12.81	9	10.36	0	-14.77
0	13	4	0	9	0	0	0
-0.7	4.6	4	-3.99	9	-6.44	0	2.03

Table 1: Correlative and structural terms of the variance decomposition for the example. (Borgonovo, Plischke, R., 202+)

U.S. airlines costs data

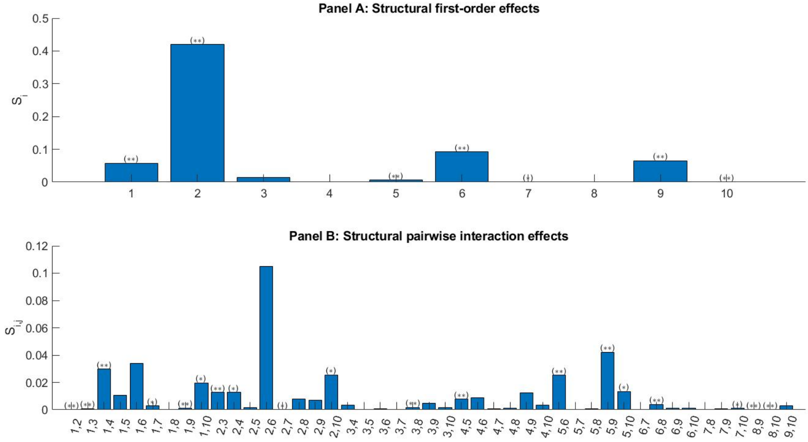


Figure 1: From Borgonovo, Plischke, R. (202+)

Structural total effects

Definition

We define the structural total importance index as

$$T_u = \sum_{v : v \cap u \neq \emptyset} V_v$$

for all input groups $u \subseteq N$.

Theorem

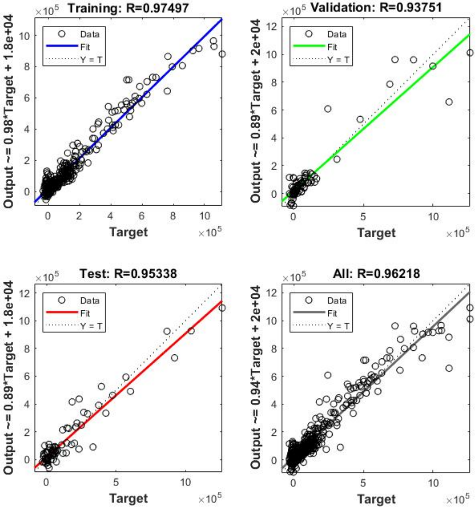
A consistent estimator of T_u is

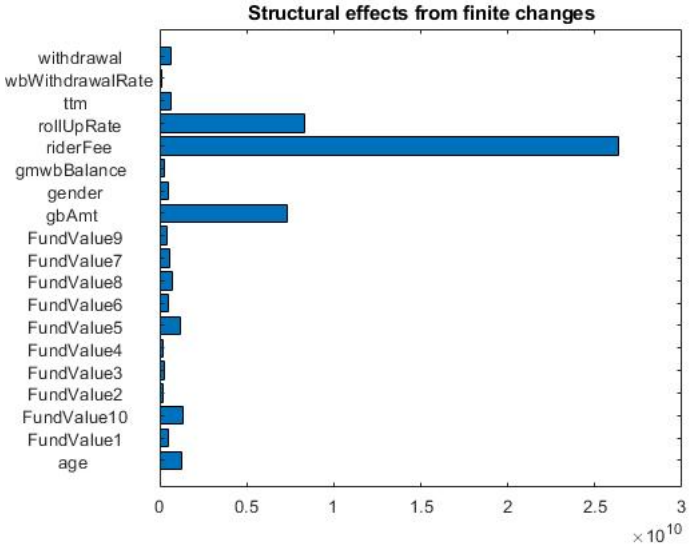
$$T_u = \frac{1}{2}\mathbb{E}\left[\phi_u^{X^0 \to X^!}\right]^2$$

where $\phi_u^{X^0 \to X^!}$ is the term of the finite-change decomposition

$$f(X^1) - f(X^0) = \sum_{i=1}^{d} \phi_i^{X^0 \to X^!} + \sum_{i<j} \phi_{i,j}^{X^0 \to X^!} + \ldots + \phi_N^{X^0 \to X^!}.$$

Variable Annuities Dataset (Gan, Valdez, 2017)




Structural effects from finite changes
withdrawal
wbWithdrawalRate
ttm
rollUpRate
riderFee
gmwbBalance
gender
gbAmt
FundValue9
FundValue7
FundValue8
FundValue6
FundValue5
FundValue4
FundValue3
FundValue2
FundValue10
FundValue1
age
0
0.5
1
1.5
2
2.5
3
$\times 10^{10}$