# Censored Regression

## Using a Multi-Task approach

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# Machine Learning Methods

## And their Issues

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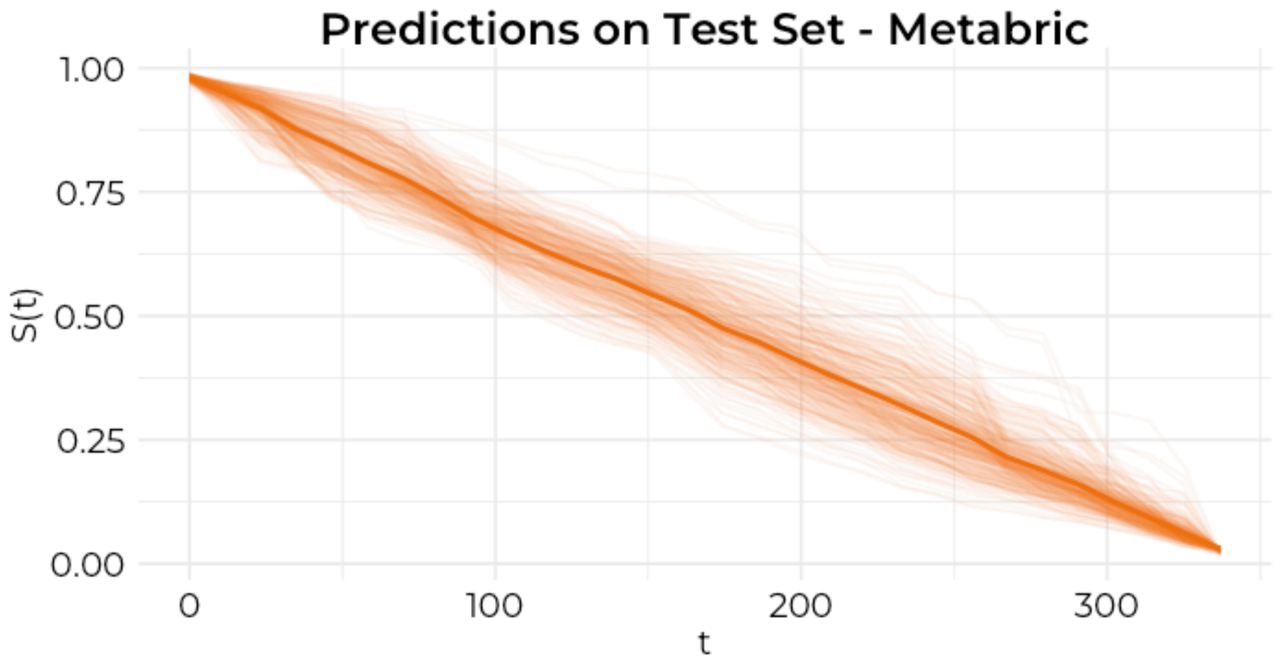

# Machine Learning Methods

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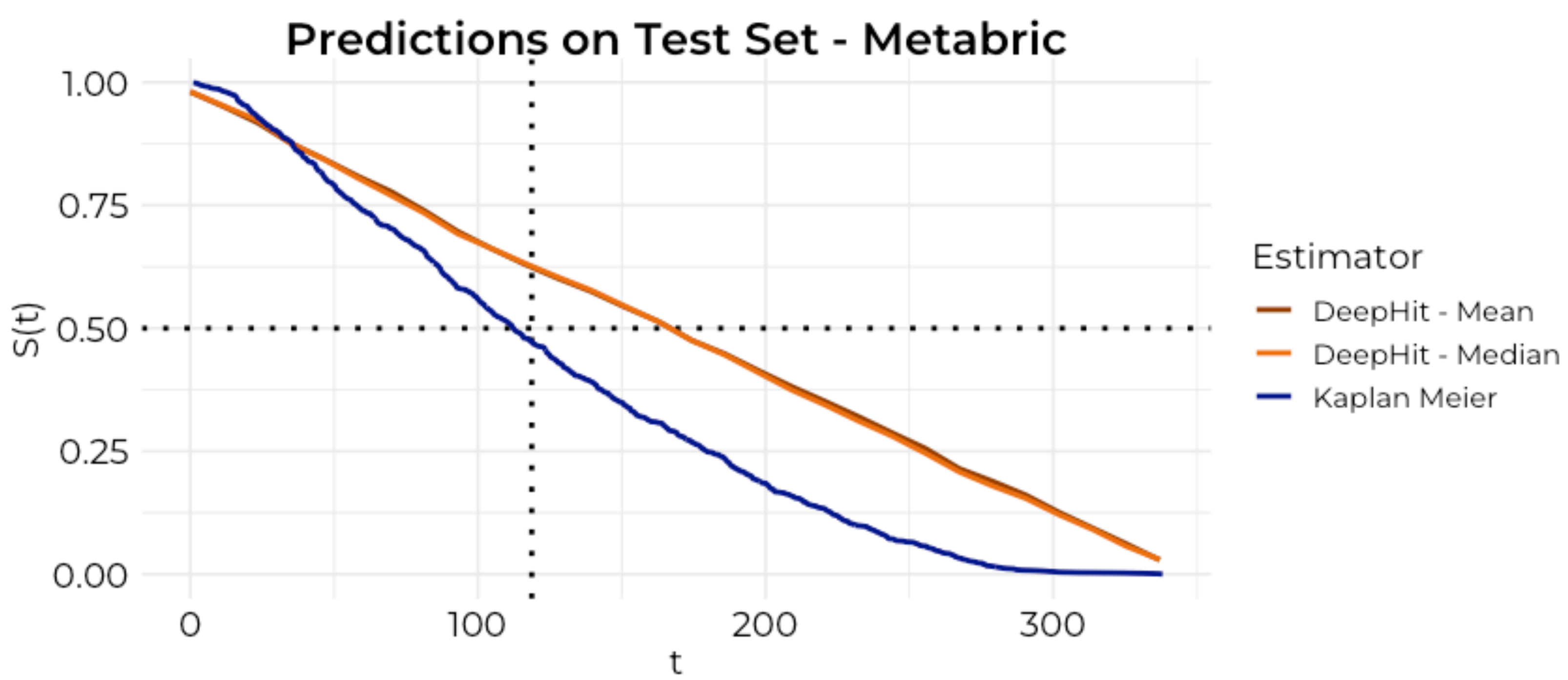
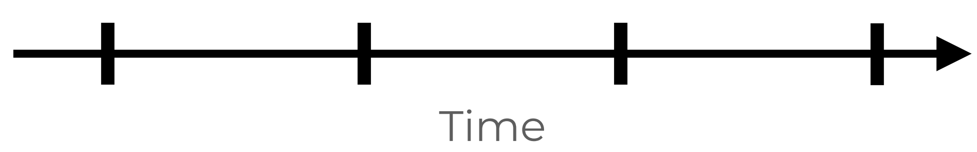


# Multi-Task Approach

- Pioneered by Yu et al. (2011). If we have tools to handle binary predictions, we can extend this to reformulate common survival problems
- Instead of directly modelling survival, consider a simple model for $z_j = \mathbb{P}(T \geq t_j | x)$



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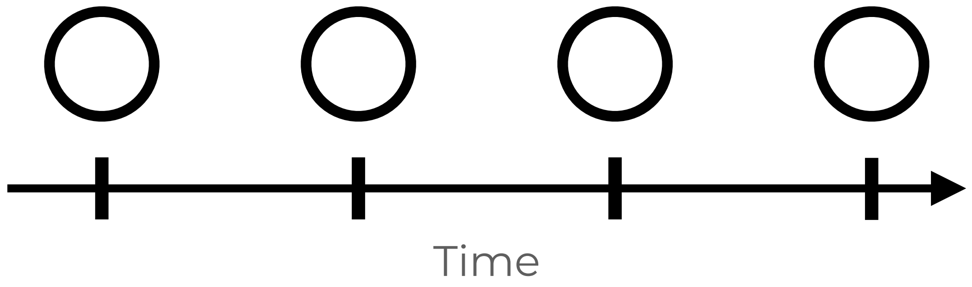

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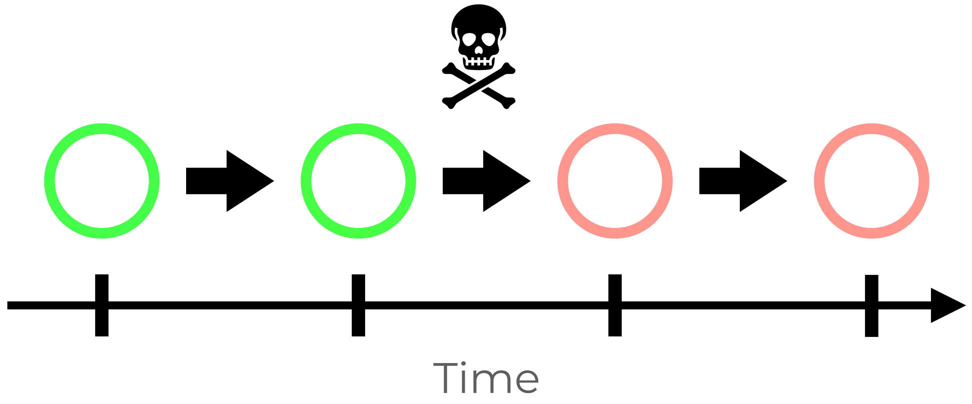
- Instead of directly modelling survival, consider a simple model for $z_j = \mathbb{P}(T \geq t_j | x)$
- But now construct a series of dependent regression tasks instead



# Conditioned Kaplan-Meier

## Setup - II

- We need to consider censored instances

- Consider a weighting scheme creating a vector (or multi-task) estimation problem

$$Y_{i,j} = \begin{cases} 0 \text{ if } \tau_i < j \\ 1 \text{ otherwise} \end{cases} \forall i, j = 0,1,\ldots,K$$

$$W_{i,j} = \begin{cases} 0 \text{ if } c_i < j \\ 1 \text{ otherwise} \end{cases} \forall i, j = 0,1,\ldots,K$$

$\tilde{\tau} = 4 \quad [1,1,1,1,0,\ldots,0]$

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- Which yields the likelihood estimator(s)

$$\hat{z}_1 = \arg\max_{z_1} \prod_{i=1}^{n} z_1^{w_{i,1}y_{i,1}}(1 - z_1)^{w_{i,1}(1-y_{i,1})} \quad \cdots \quad \hat{z}_j = \arg\max_{z_j} \prod_{i=1}^{n} z_j^{w_{i,j}y_{i,j}}(1 - z_j)^{w_{i,j}(1-y_{i,j})}$$

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## Setup - III

- Also need some restrictions (as in the original Kaplan-Meier estimation)

$$S(t_j) = 1 - \mathbb{P}[\tau = t_j \,|\, \tau \geq t_j] S(t_{j-1}) = \prod_{l=1}^{j} [1 - h(t_l)]$$

- Here we simply impose directly:

$$\hat{z}_j = \begin{cases} \hat{q}(t_1) & \text{if } j = 1 \\ \hat{z}_{j-1}\hat{q}(t_j) & \text{if } j > 1 \end{cases}$$

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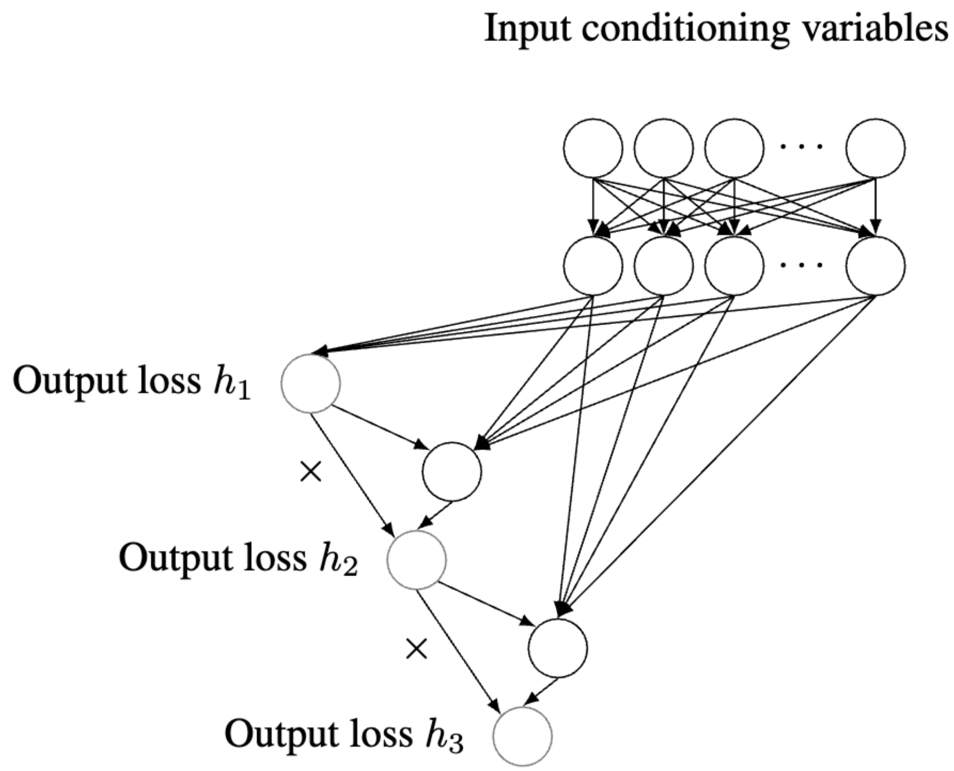
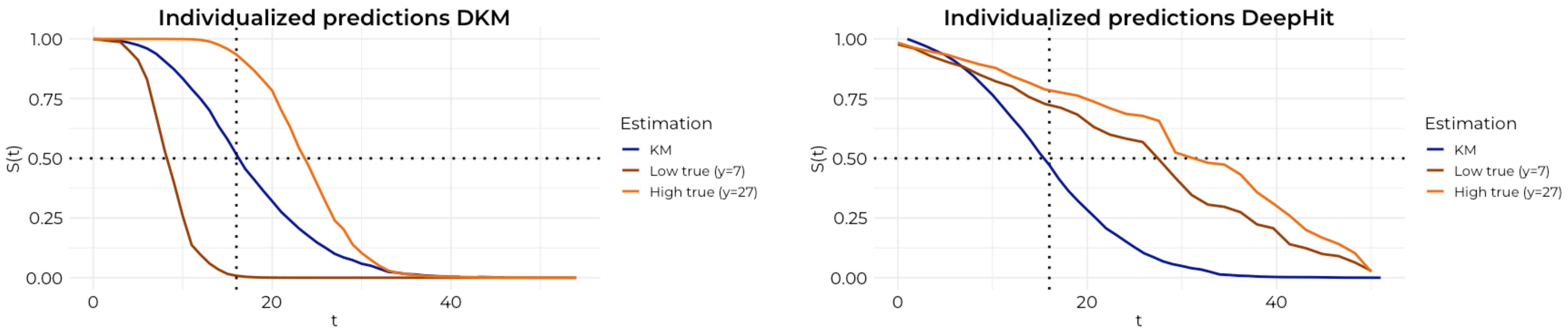


# Deep Kaplan-Meier

## Individual Predictions

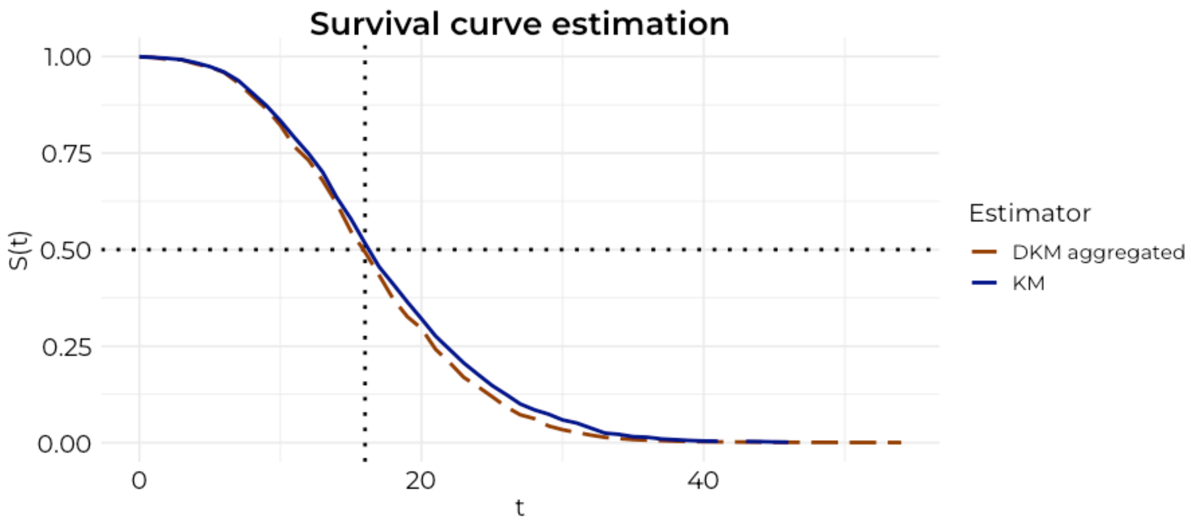
- This allows to construct conditional predictions, without assumptions such as proportional hazards
- Here: a simple example where $\tau_i = \mathcal{G}(x_i^\top \beta, 1)$ and censoring is random

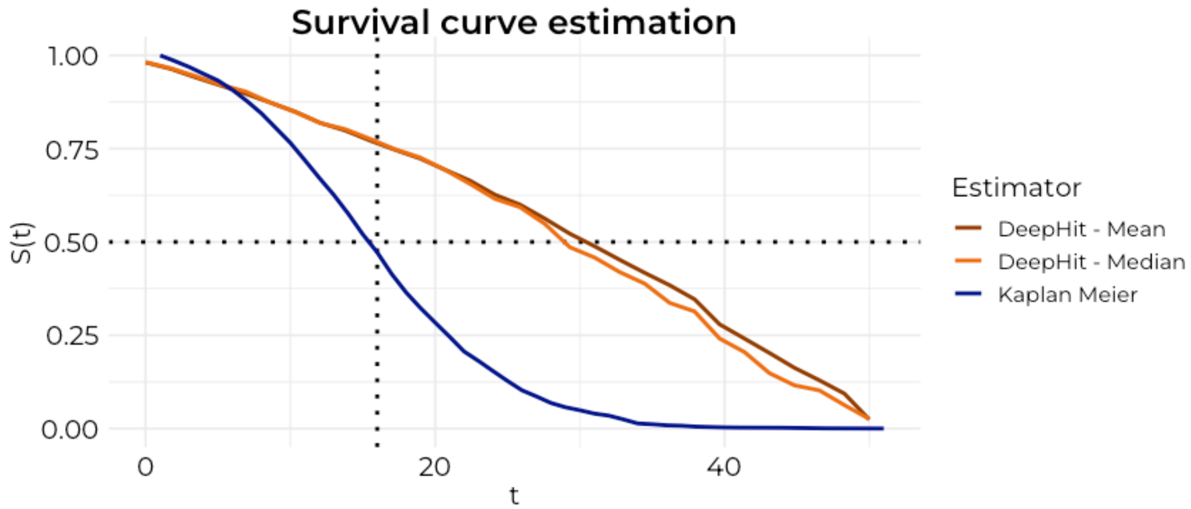


# Deep Kaplan-Meier

## Averaged Predictions

- But what about (average) calibration? $\mathbb{E}[Y \mid \hat{m}(\boldsymbol{X})] = \hat{m}(\boldsymbol{X})$
- Here: The average prediction



# Deep Kaplan-Meier

## Random Censoring

- Optimisation is straightforward, unlike in the Cox-Family
- Further, we can show that in expectation, the estimation converges to the Kaplan-Meier estimation
- The estimator also converges if event-time is only conditionally independent of the censoring time

# Deep Kaplan-Meier

## Dependent Censoring

- Consider the case where we have a positive dependence, that is $\mathbb{P}[c_i] = 1 - \min\left\{2 \times \left(\frac{x_i^\top \beta}{\max(x_i^\top \beta)}\right), 1\right\}$
- Then $\mathbb{E}[z_j] = \mathbb{E}[\hat{z}_j] + \frac{1}{\mathbb{E}[w]}\text{Cov}(w, y)$

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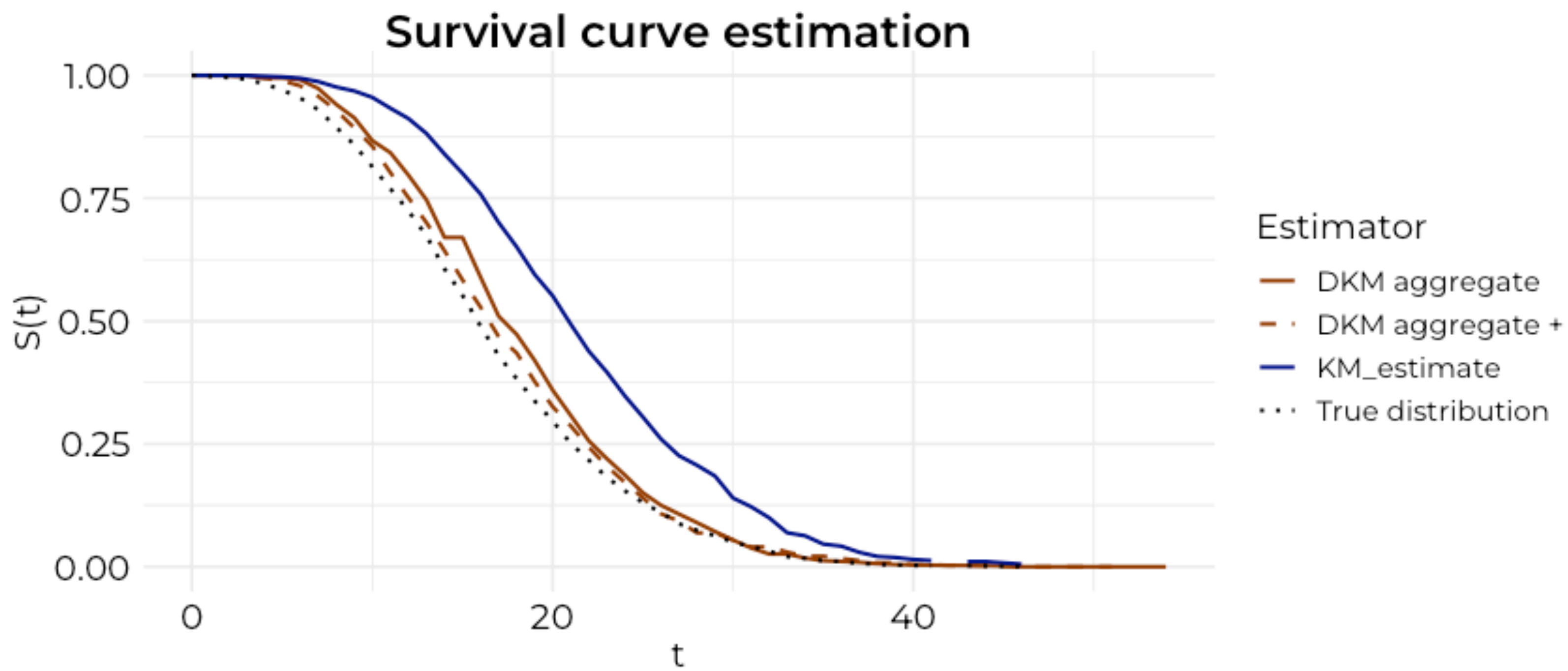

# In Summary

- Imposing structure into Neural Networks allows to:
    - Have calibrated outputs
    - Safeguards on the estimation allow usage even on small datasets
- Nonlinearities and *conditional* independence enable more realistic estimations
- Many extensions possible, on Quantiles, with included censoring model, etc..

# References


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