STOCHASTIC PROGRAMMING FOR ASSET ALLOCATION IN PENSION FUNDS

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JOINT WORK WITH JOËL WAGNER

```
> city <- "Paris"
> date <- as.Date("2017-06-08")
```

INTRODUCTION

Common approaches for asset allocation / ALM in pension funds:

- Immunization methods
- Asset optimization
- Surplus optimization
- Liability-driven investment strategies
- Stochastic control
- Stochastic programming (SP)
- Monte-Carlo simulation methods (MC)

RESEARCH PUPOSES

[Wiki]: Stochastic programming (SP) is a framework for modeling optimization problems that involve uncertainty.

Purposes:

- Review possible models
- Build a scalable model (in R)
- Analyze the convergence
- Analyze the sensitivity
- Compare the performance of the SP approach with MC methods

EXAMPLE OF SP

J.R. Birge and F. Louveaux
Introduction to Stochastic Programming, p. 21

Problem framework:

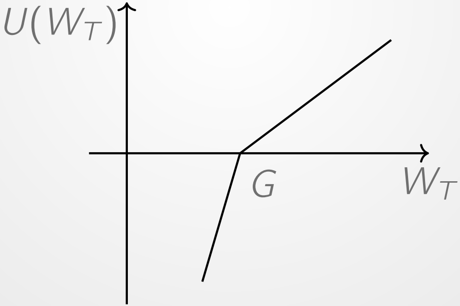
- $T = 2$: planning horizon
- $W_0 = 55$: initial wealth
- $G = 80$: target wealth
- Two asset classes available for investment

Problem: find the optimal asset allocation
Challenge: stochastic returns

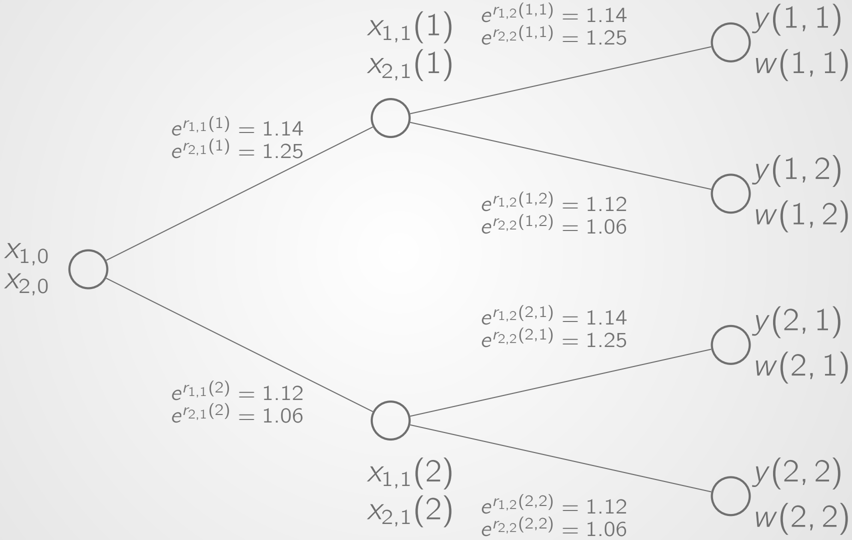
EXAMPLE OF SP

(Linear) utility function:

- $U(W_T) = q \cdot (W_T - G)^+ - r \cdot (G - W_T)^+$
- $q = 1$: surplus reward
- $r = 4$: shortage penalty



EXAMPLE OF SP



EXAMPLE OF SP

$$\max \sum_{s_1=1}^{2} \sum_{s_2=1}^{2} \frac{1}{4} \cdot (y(s_1, s_2) - 4 \cdot w(s_1, s_2))$$

$$s. \quad t. \quad x_{1,0} + x_{2,0} = 55$$

$$1.14 \cdot x_{1,0} + 1.25 \cdot x_{2,0} - x_{1,1}(1) - x_{2,1}(1) = 0$$

$$1.12 \cdot x_{1,0} + 1.06 \cdot x_{2,0} - x_{1,1}(2) - x_{2,1}(2) = 0$$

$$1.14 \cdot x_{1,1}(1) + 1.25 \cdot x_{2,1}(1) - y(1,1) + w(1,1) = 80$$

$$1.12 \cdot x_{1,1}(1) + 1.06 \cdot x_{2,1}(1) - y(1,2) + w(1,2) = 80$$

$$1.14 \cdot x_{1,1}(2) + 1.25 \cdot x_{2,1}(2) - y(2,1) + w(2,1) = 80$$

$$1.12 \cdot x_{1,1}(2) + 1.06 \cdot x_{2,1}(2) - y(2,2) + w(2,2) = 80$$

$$x \geq 0, y \geq 0, w \geq 0$$

POSSIBLE MODELS

Objective function:

- Maximize the total value of assets
- Maximize the expected value of the utility
- Maximize the funding ratio
- Minimize the contribution rate or the capital injection, etc.

Risk constraints:

- Chance constraints (ruin probability)
- Integrated chance constraints (TVaR)

Optimize values:

- At the final nodes
- Also at intermediate nodes

UNDERLYING ECONOMIC MODEL

Vector-autoregressive model (of order p in matrix form):

$$\boldsymbol{r}_t = \boldsymbol{m} + \Theta_1 \boldsymbol{r}_{t-1} + \Theta_2 \boldsymbol{r}_{t-2} + \ldots + \Theta_p \boldsymbol{r}_{t-p} + \boldsymbol{\epsilon}_t, \quad (1)$$

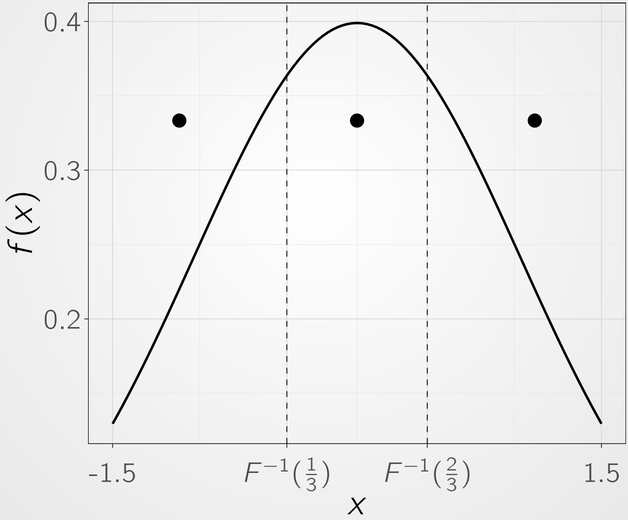
Example of VAR(2) for two assets:

$$r_{1,t} = m_1 + \theta_{1,1} \cdot r_{1,t-1} + \theta_{1,2} \cdot r_{2,t-1} + \epsilon_{1,t}$$
$$r_{2,t} = m_2 + \theta_{2,1} \cdot r_{1,t-1} + \theta_{2,2} \cdot r_{2,t-1} + \epsilon_{2,t}$$

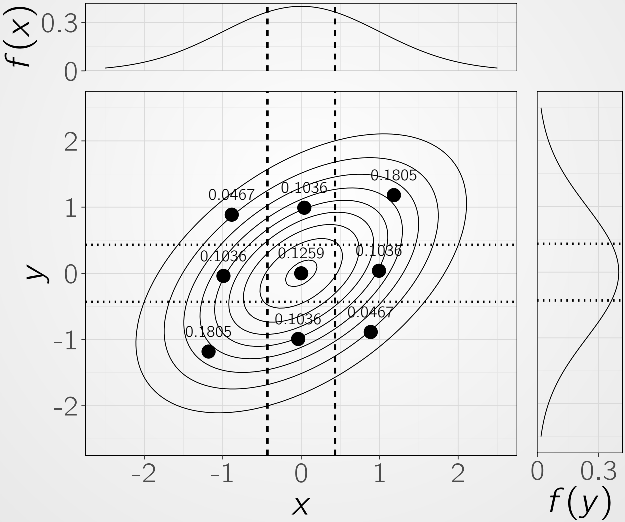
SCENARIO TREE GENERATION METHODS

- Sampling methods
- "Bracket-mean" and "bracket-median"
- Moment matching method via integration quadratures
- "Optimal discretization"
- Other more exotic methods

"BRACKET-MEAN" FOR UNIVARIATE $N(0,1)$ AND $k=3$



"BRACKET-MEAN" FOR BIVARIATE NORMAL DISTRIBUTION ($\rho = 0.5$)



IMPLEMENTATIONAL DETAILS (R SIDE)

- Packages for analyzing time series: `vars`, `het.test`
- Packages for multidimensional integration: `cubature`, `R2Cuba`
- Solver packages: `linprog`, `lpSolve` (wrapper for `lp_solve`), `Rglpk` (wrapper for `GLPK`)

CURRENT SOLUTION

The routine is controlled by Shell script, which execute:

- R script: calibrate the VAR model
- R script: generate the scenario tree
- R script: generate the problem file of CPLEX LP format
- `glpsol` command: process such files and solve the LP problem

CONVERGENCE & SENSITIVITY ANALYSIS

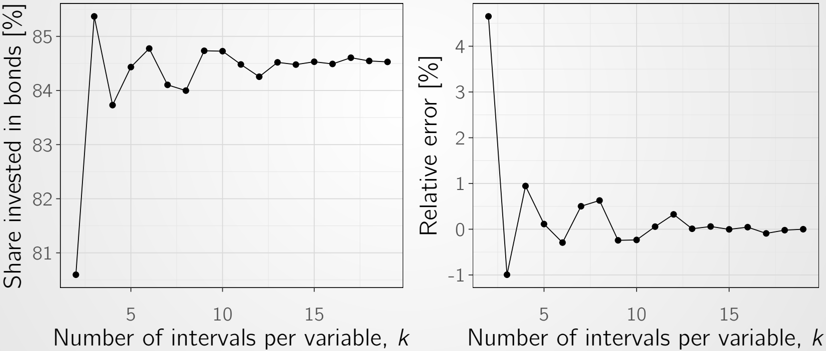
Investigate and study:

- Convergence of the optimal solution with respect to the number of intervals per variable k
- Sensitivity of the optimal solution to changes in parameters of the model

Key performance indicators:

- Initial allocation
- Probability of excess
- Probability of deficit
- Mean of surplus given excess
- Mean of shortage given deficit

CONVERGENCE ANALYSIS (EXAMPLE)



SENSITIVITY ANALYSIS

- Planning horizon T
- Target wealth L_T
- Shortage penalty r
- Bond's mean return m_{bonds}
- Volatility of stocks' residuals $\sigma_{\text{stocks},t}$

MONTE CARLO

- Simulate $N = 10000$ paths of VAR model.
- Fix the initial asset allocation at $t = 0$. Using "Buy&Hold" strategy calculate the final wealth for each of the simulated path.
- Estimate quantities of interest.

RESEARCH SUMMARY

We have been studied:

- Various scenario tree generation techniques
- Possible software and solvers
- The convergence of the optimal solution with respect to the bushiness of the scenario tree
- The relation between the optimal solution and model's characteristics (planning horizon T, target wealth L_T, etc)

Possible extensions:

- More sophisticated economic models
- Stochastic liability part
- Implement regulatory constraints

THANK YOU!