STOCHASTIC PROGRAMMING FOR ASSET ALLOCATION IN PENSION FUNDS

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> city <- "Paris"
> date <- as.Date("2017-06-08")
INTRODUCTION

Common approaches for asset allocation / ALM in pension funds:

- Immunization methods
- Asset optimization
- Surplus optimization
- Liability-driven investment strategies
- Stochastic control
- Stochastic programming (SP)
- Monte-Carlo simulation methods (MC)
RESEARCH PURPOSES

[Wiki]: Stochastic programming (SP) is a framework for modeling optimization problems that involve uncertainty. Purposes:

- Review possible models
- Build a scalable model (in R)
- Analyze the convergence
- Analyze the sensitivity
- Compare the performance of the SP approach with MC methods
EXAMPLE OF SP

J.R. Birge and F. Louveaux

*Introduction to Stochastic Programming*, p. 21

Problem framework:
- \( T = 2 \): planning horizon
- \( W_0 = 55 \): initial wealth
- \( G = 80 \): target wealth
- Two asset classes available for investment

Problem: find the optimal asset allocation
Challenge: stochastic returns
EXAMPLE OF SP

(Linear) utility function:

- \( U(W_T) = q \cdot (W_T - G)^+ - r \cdot (G - W_T)^+ \)
- \( q = 1 \): surplus reward
- \( r = 4 \): shortage penalty
EXAMPLE OF SP

$$\max \sum_{s_1=1}^{2} \sum_{s_2=1}^{2} \frac{1}{4} \cdot (y(s_1, s_2) - 4 \cdot w(s_1, s_2))$$

s. t.  \(x_{1,0} + x_{2,0} = 55\)

1.14 \(x_{1,0} + 1.25 \cdot x_{2,0} - x_{1,1}(1) - x_{2,1}(1) = 0\)

1.12 \(x_{1,0} + 1.06 \cdot x_{2,0} - x_{1,1}(2) - x_{2,1}(2) = 0\)

1.14 \(x_{1,1}(1) + 1.25 \cdot x_{2,1}(1) - y(1, 1) + w(1, 1) = 80\)

1.12 \(x_{1,1}(1) + 1.06 \cdot x_{2,1}(1) - y(1, 2) + w(1, 2) = 80\)

1.14 \(x_{1,1}(2) + 1.25 \cdot x_{2,1}(2) - y(2, 1) + w(2, 1) = 80\)

1.12 \(x_{1,1}(2) + 1.06 \cdot x_{2,1}(2) - y(2, 2) + w(2, 2) = 80\)

\(x \geq 0, y \geq 0, w \geq 0\)
POSSIBLE MODELS

Objective function:
- Maximize the total value of assets
- Maximize the expected value of the utility
- Maximize the funding ratio
- Minimize the contribution rate or the capital injection, etc.

Risk constraints:
- Chance constraints (ruin probability)
- Integrated chance constraints (TVaR)

Optimize values:
- At the final nodes
- Also at intermediate nodes
Vector-autoregressive model (of order $p$ in matrix form):

$$ r_t = m + \Theta_1 r_{t-1} + \Theta_2 r_{t-2} + \ldots + \Theta_p r_{t-p} + \epsilon_t, \quad (1) $$

Example of VAR(2) for two assets:

$$ r_{1,t} = m_1 + \theta_{1,1} \cdot r_{1,t-1} + \theta_{1,2} \cdot r_{2,t-1} + \epsilon_{1,t} $$

$$ r_{2,t} = m_2 + \theta_{2,1} \cdot r_{1,t-1} + \theta_{2,2} \cdot r_{2,t-1} + \epsilon_{2,t} $$
SCENARIO TREE GENERATION METHODS

• Sampling methods
• "Bracket-mean" and "bracket-median"
• Moment matching method via integration quadratures
• "Optimal discretization"
• Other more exotic methods
"BRACKET-MEAN" FOR UNIVARIATE $N(0, 1)$ AND $k = 3$
"BRACKET-MEAN" FOR BIVARIATE NORMAL DISTRIBUTION ($\rho = 0.5$)
IMPLEMENTATIONAL DETAILS (R SIDE)

- Packages for analyzing time series: \texttt{vars}, \texttt{het.test}
- Packages for multidimensional integration: \texttt{cubature}, \texttt{R2Cuba}
- Solver packages: \texttt{linprog}, \texttt{lpSolve} (wrapper for \texttt{lp_solve}), \texttt{Rglpk} (wrapper for \texttt{GLPK})
CURRENT SOLUTION

The routine is controlled by Shell script, which execute:

- R script: calibrate the VAR model
- R script: generate the scenario tree
- R script: generate the problem file of CPLEX LP format
- `glpsol` command: process such files and solve the LP problem
CONVERGENCE & SENSITIVITY ANALYSIS

Investigate and study:

- Convergence of the optimal solution with respect to the number of intervals per variable $k$
- Sensitivity of the optimal solution to changes in parameters of the model

Key performance indicators:

- Initial allocation
- Probability of excess
- Probability of deficit
- Mean of surplus given excess
- Mean of shortage given deficit
CONVERGENCE ANALYSIS (EXAMPLE)

Number of intervals per variable, $k$

Share invested in bonds [%]

Relative error [%]

Number of intervals per variable, $k$
SENSITIVITY ANALYSIS

- Planning horizon \( T \)
- Target wealth \( L_T \)
- Shortage penalty \( r \)
- Bond’s mean return \( m_{\text{bonds}} \)
- Volatility of stocks’ residuals \( \sigma_{\text{stocks}, t} \)
MONTE CARLO

- Simulate $N = 10000$ paths of VAR model.
- Fix the initial asset allocation at $t = 0$. Using “Buy&Hold" strategy calculate the final wealth for each of the simulated path.
- Estimate quantities of interest.
RESEARCH SUMMARY

We have been studied:

- Various scenario tree generation techniques
- Possible software and solvers
- The convergence of the optimal solution with respect to the bushiness of the scenario tree
- The relation between the optimal solution and model’s characteristics (planning horizon $T$, target wealth $L_T$, etc)

Possible extensions:

- More sophisticated economic models
- Stochastic liability part
- Implement regulatory constraints
THANK YOU!