Computing Capital Requirements with Guarantees

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Asset-Liability Risk

0 = current time τ = risk horizon (e.g. 1 year)

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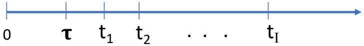

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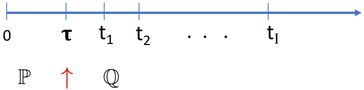
Pasting together the *real world measure* $\mathbb{P}$ **and the** *pricing measure* $\mathbb{Q}$



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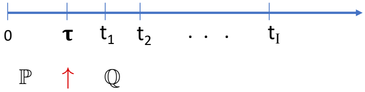

$$\mathbb{P} \otimes \mathbb{Q}[A] = \int_{\mathbb{R}^d} \mathbb{Q}[A \mid X = x] \pi(dx), \quad A \in \mathcal{F},$$

where π is the distribution of $X = (X_1, \ldots, X_d)$

Conditional Expectations as Minimizing Functions

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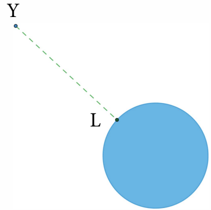

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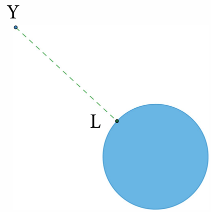

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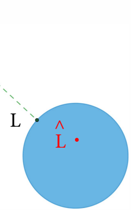

- **Least Squares Monte Carlo**

simulate (X^j, Y^j) and solve $\min_{f\in\mathcal{S}} \frac{1}{J}\sum_{j=1}^{J}\left(Y^j - f(X^j)\right)^2$

over a subfamily $\mathcal{S}$ of all Borel functions $f\colon \mathbb{R}^d \to \mathbb{R}$.

Least Squares Regression

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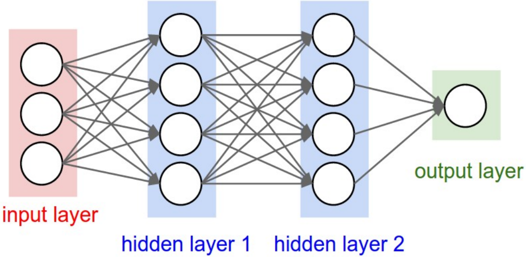
- regression trees Boudabsa and Filipović (2022)
- neural network regression Kohlen et al (2010), Fiore et al. (2018), Cheridito et al. (2020)

Here, we minimize

$$\theta \mapsto \frac{1}{J} \sum_{j=1}^{J} \left(Y^j - f_\theta(X^j)\right)^2$$

over a set of *neural networks* $f_\theta : \mathbb{R}^d \to \mathbb{R}, \quad \theta \in \mathbb{R}^q$



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- Let $X^1, \ldots, X^n$ be *independent $\mathbb{P}$-simulations* of $X = (X_1, \ldots, X_d)$

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$$L^{(1)} = f_\theta(X^{(1)}) \geq \ldots \geq L^{(n)} = f_\theta(X^{(n)})$$

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there exist convergence rates

see. e.g., David and Nagaraja (2003) and Zwingmann and Holzmann (2016)

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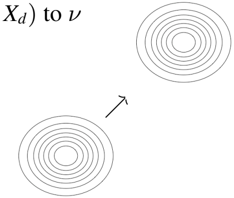

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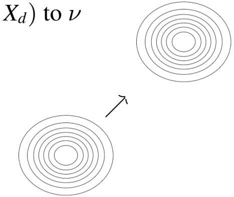

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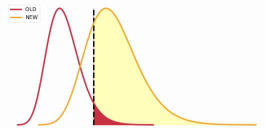

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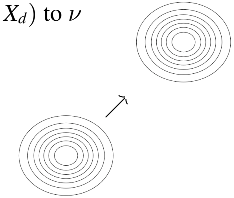

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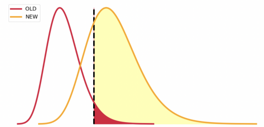

- note that $f = \arg\min_{\varphi\,:\,\mathbb{R}^d \to \mathbb{R}} \mathbb{E}^{\mathbb{P}^\nu \otimes \mathbb{Q}}\left[(Y - \varphi(X))^2\right]$

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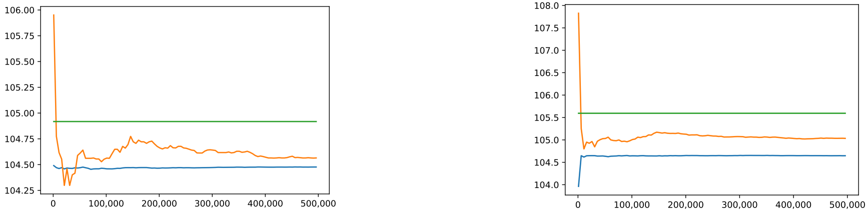

empirical 99.5%-VaR

empirical 99%-ES

without IS with IS reference values obtained from Black–Scholes

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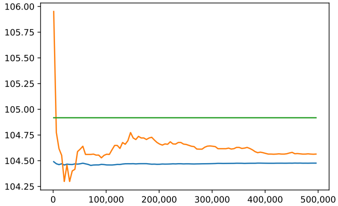

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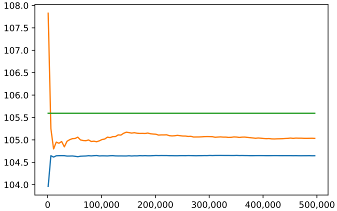

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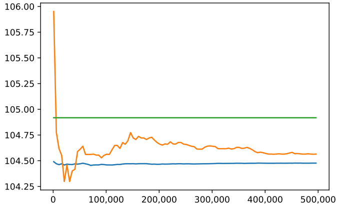

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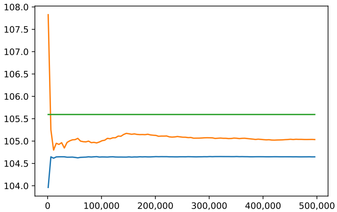

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Approximation Error I: $L = f(X) \approx \hat{L} = f_\theta(X)$

Approximation Error II: $\mathrm{ES}_\alpha(\hat{L}) \approx \widehat{\mathrm{ES}}_\alpha(\hat{L})$

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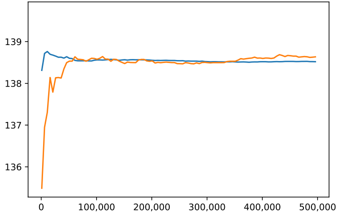

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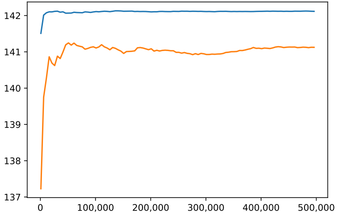

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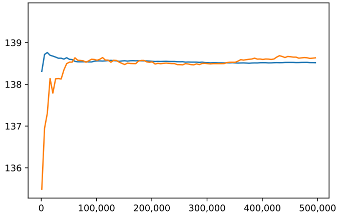
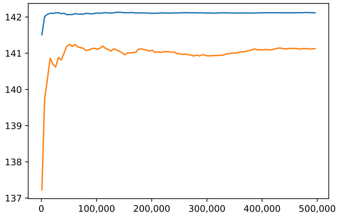
$$L = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_\tau^T r_s + \mu_{x+s} ds} \max\left\{e^{q_T}, b\,a_{x+T}(T)\right\} \mid q_\tau,\ r_\tau,\ \mu_{x+\tau}\right]$$

where $a_{x+T}(T) =$ time-T value of a life-time annuity and

$q_\tau =$ log-stock index, $r_\tau =$ interest rate, $\mu_{x+\tau} =$ mortality rate



empirical 99.5%-VaR



empirical 99%-ES

without IS with IS

Approximation Error I: $L = f(X) \approx \hat{L} = f_\theta(X)$ *black box!*

Example 2: Variable Annuity with GMIB (Guaranteed Minimum Income Benefit)

(Ha and Bauer, 2021)

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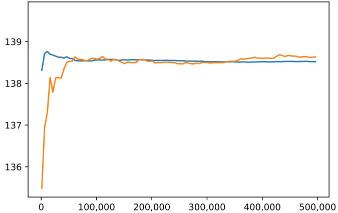

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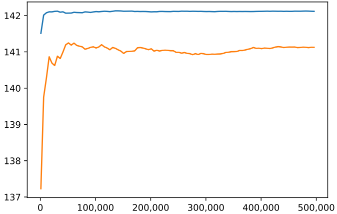

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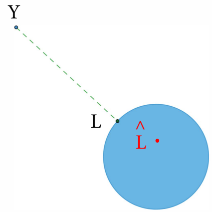
Approximation Error I: $L = f(X) \approx \hat{L} = f_\theta(X)$ *black box!*

Approximation Error II: $\mathrm{ES}_\alpha(\hat{L}) \approx \widehat{\mathrm{ES}}_\alpha(\hat{L})$ *well understood*

Computation of Conditional Expectations with Guarantees

Goal   Derive an *alternative representation* of the minimal L^2-distance



Computation of Conditional Expectations with Guarantees

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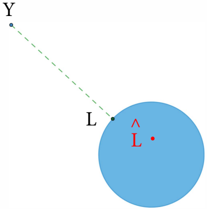
$$\min_{f\,:\,\mathbb{R}^d \to \mathbb{R}} \|Y - f(X)\|_2 = \|Y - \mathbb{E}[Y \mid X]\|_2$$



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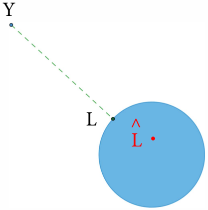

Assumption Y is of the form $Y = h(X, V)$ for a known function $h : \mathbb{R}^{d+k} \to \mathbb{R}$ and a k-dim random vector V independent of X

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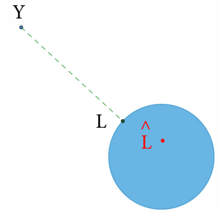

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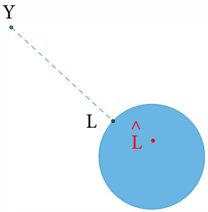

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Theorem $\|Y - \mathbb{E}[Y \mid X]\|_2^2 = \mathbb{E}[Y(Y - Z)]$

- For any *candidate regression function* $\hat{f}\colon \mathbb{R}^d \to \mathbb{R}$ one has

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$\uparrow$ $\nwarrow$ $\nearrow$

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↑ ↖ ↗

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- Problem: for $\alpha = 0.99$, $\frac{1}{1-\alpha} = 100$

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- *We assume* $\left\|L - \hat{L}\right\|_{L^2(\mathbb{P}_\alpha)} \approx \left\|L - \hat{L}\right\|_{L^2(\hat{\mathbb{P}}_\alpha)}$

Numerical Results

	$\widehat{\mathrm{ES}}_\alpha(\hat{L})$	error	relative error
Option Portfolio	104.6	$\pm$ 1.5	$\pm$ 1.4 %
Variable Annuity	142.0	$\pm$ 1.7	$\pm$ 1.2 %

Thank You!